The Field of Reals is not ω-Automatic

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Abstract

We investigate structural properties of ω-automatic presentations of infinite structures in order to sharpen our methods to determine whether a given structure is ω-automatic. We apply these methods to show that no field of characteristic 0 admits an injective ω-automatic presentation, and that uncountable fields with a definable linear order cannot be ω-automatic.

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1 Introduction

Automatic structures are (in general) infinite structures that admit a finite presentation by automata. Informally, an automatic presentation of a relational structure $\mathfrak{B} = (B, R_1, \ldots, R_m)$ consists of a language L, which must be recognisable by an automaton $\mathcal{A}$, and a surjective function $\pi : L \to B$ that associates every word of L with the element of B that it represents. The function π must be surjective (every element of B is named by some word in L) but need not be injective (elements may have more than one name). In addition it must be recognisable by automata, reading their inputs synchronously, whether two elements of L name the same element of B, and, for each relation R_i, whether a given tuple of words in L names a tuple in R_i. Together, the automaton $\mathcal{A}$ and the automata that recognise equality and the relations $R_1, \ldots, R_m$ provide a finite representation of the structure $\mathfrak{B}$. This concept has implicitly been known since the first days of automata theory, but the systematic investigation of such presentations was mainly invoked by the work of Khoussainov and Nerode [7] and later by the work of Blumensath and Grädel [1].

In principle we can use automata over finite words, infinite words, finite trees, or infinite trees to obtain different classes of automatic structures. Indeed, any model of automata can be used for this approach as long as it is effectively closed under all first-order operations (union, intersection, complementation, and projection) and its emptiness problem is decidable.

As a consequence of these two properties it follows that

- every automatic structure has a decidable first-order theory and, more generally,
- given any automatic presentation of $\mathfrak{A}$ and any first-order formula $\varphi(x_1, \ldots, x_k)$ one can effectively construct an automaton that represents the relation $\varphi^{\mathfrak{A}} = \{\bar{a} \in A^k : \mathfrak{A} \models \varphi(\bar{a})\}$.

Thus, all definable properties of automatic structures can be algorithmically investigated using automata-theoretic methods based on appropriate finite presentations. This makes automatic structures a domain of considerable interest for computer science.

While the case of word-automatic structures (with presentations based on automata operating on finite words) is reasonably well understood, much less is known for presentations based on other classes of automata. In particular, our methods to establish whether a given structure admits an automatic presentation on, say, infinite words, or to classify all such structures inside a given domain, are still relatively weak. For countable structures, an important achievement is the result by Kaiser, Rubin, and Bárány [6] that a countable structure is ω-automatic if, and only if, it is word-automatic. Thus the recent result by Tsankov [14] that the additive group of the rationals is not automatic immediately implies that it is not ω-automatic either. Another interesting result for word-automatic structures is due to Khoussainov, Nies, Rubin and Stephan. In [8] among others they show that there are no infinite automatic fields.

One of the most prominent and important structures with a decidable first-order theory is certainly the field of reals $(\mathbb{R}, +, \cdot)$. The decidability goes back to Tarski [13] and is based on a quantifier elimination argument. Therefore, it is very natural to ask whether the field of reals admits an automatic presentation. Of course, such a presentation cannot be based on automata on finite words (or finite trees) because languages of finite words and trees are countable. However, it might be the case that the field of reals is ω-automatic, i.e., admits a presentation based on automata on infinite words, or that it is ω-tree-automatic, with a presentation based on automata on infinite trees.

The question whether this is the case is closely related to classical problems raised by Büchi and Rabin in the context of decidable first-order theories. The decidability of Presburger arithmetic, the first-order theory of $(\mathbb{N}, +)$, had originally been proved by quantifier elimination, but Büchi's automata-based proof of the decidability of S1S (the monadic theory of infinite words) easily carries over to an automata-theoretic decidability argument for Presburger arithmetic. And indeed, $(\mathbb{N}, +)$ is now one of the standard examples for word-automatic structures. In Rabin's classical paper [11], where he proved the decidability of S2S (the monadic theory of the infinite binary tree) and several other theories, he explicitly raised the question whether also the decidability of the field of reals could be proved by automata-theoretic methods.

In this paper we investigate the question whether the field of reals, and other fields of characteristic 0, are ω-automatic. It is quite easy to see that both reducts $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$ of the field of reals are ω-automatic but it has so far been open whether two such presentations could be combined to one of the entire field. We shall prove that this is not the case. More generally, we establish the following results.

▶ **Theorem 1.** *No field of characteristic 0 admits an injective ω-automatic presentation.*

▶ **Theorem 2.** *No field of characteristic 0 with a definable linear order is ω-automatic.*

In particular, the field of reals does not admit an ω-automatic presentation. Notice that this does not completely settle Rabin's question, since it might still be the case that the field of reals is ω-tree-automatic. We do not expect this, but our current methods do not seem to be powerful enough to settle this question.

We briefly outline our methodology. We shall make heavy use of the notion of *end-equivalence* of infinite words: two words $v, w \in \Sigma^\omega$ are end-equivalent, in short $v \sim_e w$, if they are equal from some position onwards, i.e. $v[n, \omega) = w[n, \omega)$ for some $n \in \mathbb{N}$, and we analyse the size of $\sim_e$-equivalent sets in ω-automatic structures.

In the study of ω-automatic presentations, it is important to distinguish between injective and non-injective presentations. Injective presentations are much easier to work with, and, of course, they have the advantage that we do not need an automaton to determine whether two words encode the same elements. In the case of word-automatic structures it is not hard to see that we can always find an injective automatic presentation, which makes our life much easier. On the other side, it had been open for some time whether ω-automatic structures always admit injective presentations, until Hjorth, Khoussainov, Montalbán and Nies [5] were able to describe an ω-automatic structure that does not even permit an injective Borel presentation (which is a much more general notion than an injective ω-automatic presentation).

Nevertheless, many ω-automatic structures do admit injective presentations such as, for instance, the reducts $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$ of the field of reals. Therefore it is also interesting to ask what kind of structures admit an injective presentation.

Our first step will be the observation that every injective ω-automatic presentation of an infinite structure necessarily produces an infinite set of $\sim_e$-equivalent elements. On the other side, we shall prove the following general fact on ω-automatic fields.

▶ **Lemma 3.** *Whenever a field of characteristic 0 admits an ω-automatic presentation then the size of all $\sim_e$-equivalent subsets of the field is bounded by a constant.*

To establish this, we shall combine an argument saying that the image of every set of $\sim_e$-equivalent elements under a definable k-ary function can be partitioned into a small number of sets, each of which consists of $\sim_e$-equivalent elements only, with a result on additive combinatorics, known as Freiman's Theorem.

From these results, we immediately obtain Theorem 1 because on one side, an injective presentation of an infinite filed would have to contain an infinite set of $\sim_e$ equivalent elements, but on the other side, we have proved that every such set should be of bounded size.

To extend this result beyond injective presentations, more work is needed. We shall assume that our fields admit a definable linear order, and prove a strengthening of Kuske's result [9] that $(\{0,1\}^\omega, <_{lex})$ is embeddable into every ω-automatic uncountable linear order. In fact, we shall prove that every ω-automatic presentation of an uncountable linear order contains an injective automatic presentation of $(\{0,1\}^\omega, <_{lex})$. This implies that every ω-automatic presentation of an uncountable structure with a definable linear order contains an infinite $\sim_e$-equivalent set. Together with Lemma 3 this implies Theorem 2.

2 Preliminaries

First, we give a formal definition of an ω-automatic presentation.

▶ **Definition 4.** An ω-automatic presentation (over the alphabet Σ) of a structure $\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \ldots, R_n^{\mathfrak{A}})$ is a pair consisting of a structure $\mathcal{L} = (L, \approx, R_1^{\mathcal{L}}, \ldots, R_n^{\mathcal{L}})$ and a surjective function $\pi : L \to A$ such that the following holds:

- $L \subseteq \Sigma^\omega$
- $\approx = \{(v, w) : \pi(v) = \pi(w)\}$
- $R_i^{\mathcal{L}} = \{(v_1, \ldots, v_{r_i}) : (\pi(v_i), \ldots, \pi(v_{r_i})) \in R_i\}$ for $i = 1, \ldots, n$
- $L, \approx, R_1^{\mathcal{L}}, \ldots, R_n^{\mathcal{L}}$ are all ω-regular

We call a presentation injective if π is injective. In this case we will omit $\approx$ in the signature of $\mathcal{L}$. A structure is ω-automatic if it has an ω-automatic presentation.

Sometimes we are also not interested in the concrete labelling function π. In these cases we will also omit π and just demand that that $(L, R_1, \ldots, R_n)/\approx$ is isomorphic to $\mathfrak{A}$.

A relation $R \subseteq L^n$ is called ω-regular if the language

$$L_R = \{\langle v_1, \ldots, v_n\rangle \ : \ (v_1, \ldots, v_n) \in R\} \subseteq (\Sigma^n)^\omega$$

is ω-regular, where $\langle v_1, \ldots, v_n\rangle \in (\Sigma^n)^\omega$ is the so called convolution, defined by

$$\langle v_1, \ldots, v_n\rangle[i] = (v_1[i], \ldots, v_n[i]).$$

2.1 ω-Semigroups

The fundamental correspondence between recognisability by finite automata and by finite semigroups has been extended to ω-regular sets. This is based on the notion of *ω-semigroups*. Rudimentary facts on ω-semigroups are well presented in [10], we only mention what is most necessary.

An ω-semigroup $S = (S_f, S_\omega, \cdot, *, \pi)$ is a two-sorted algebra, where $(S_f, \cdot)$ is a semigroup, $* : S_f \times S_\omega \mapsto S_\omega$ is the *mixed product* satisfying, for every $s, t \in S_f$ and every $\alpha \in S_\omega$, the equality $s \cdot (t * \alpha) = (s \cdot t) * \alpha$, and where $\pi : S_f^\omega \mapsto S_\omega$ is the *infinite product* satisfying $s_0 \cdot \pi(s_1, s_2, \ldots) = \pi(s_0, s_1, s_2, \ldots)$ as well as the associativity rule:

$$\pi(s_0, s_1, s_2, \ldots) = \pi(s_0 s_1 \cdots s_{k_1}, s_{k_1+1} s_{k_1+2} \cdots s_{k_2}, \ldots)$$

for every sequence $(s_i)_{i\geq 0}$ of elements of S_f and every strictly increasing sequence $(k_i)_{i\geq 0}$ of indices. For $s \in S_f$ we write s^ω for $\pi(s, s, \ldots)$.

Morphisms of ω-semigroups are defined to preserve all three products as expected. There is a natural way to extend finite semigroups and their morphisms to ω-semigroups. As in semigroup theory, idempotents play a central role in this extension. An *idempotent* is a semigroup element $e \in S$ satisfying $ee = e$. For every element s in a finite semigroup, the sub-semigroup generated by s contains a unique idempotent s^k. The least $k > 0$ such that s^k is idempotent for every $s \in S_f$ is called the *exponent* of the semigroup S_f. Another useful notion is absorption: we say that s *absorbs* t (on the right) if $st = s$.

There is also a natural extension of the free semigroup Σ^+ to the ω-semigroup $(\Sigma^+, \Sigma^\omega)$ with $*$ and π determined by concatenation. An ω-semigroup $S = (S_f, S_\omega)$ *recognises* a language $L \subseteq \Sigma^\omega$ via a morphism $\phi : (\Sigma^+, \Sigma^\omega) \to (S_f, S_\omega)$ if $\phi^{-1}(\phi(L)) = L$. This notion of recognisability coincides with that by non-deterministic Büchi automata; constructions from Büchi automata to ω-semigroups and back are also presented in [10].

▶ **Theorem 5** ([10]).
A language $L \subseteq \Sigma^\omega$ is ω-regular if, and only if, it is recognised by a finite ω-semigroup.

3 Freiman's Theorem

On the algebraic side, we will use a result on additive combinatorics due to Freiman. It has also been an essential ingredient in Tsankov's recent proof that the additive group of rationals is not automatic [14]. Freiman's Theorem is concerned with subsets A of a group G with small doubling $A + A = \{a + a' : a, a' \in A\}$, in the sense that $|A + A| \leq c|A|$ for some constant $c > 0$. For example, consider the case $G = (\mathbb{Z}, +)$. Then, for every arithmetic progression A, we have $|A + A| < 2|A|$. For larger values of c, more complicated sets than simple arithmetic progressions are possible. To state Freiman's Theorem we need the notion of a *generalised* or *multidimensional arithmetic progression*.

▶ **Definition 6.** A d-dimensional arithmetic progression is a set P such that there exist $p_0 \in P$, $\overline{p} = (p_1, \ldots, p_d) \in P^d$ and $n_1, \ldots, n_d \in \mathbb{N}$ with

$$P = \left\{ p_0 + \sum_{i=1}^{d} m_i p_i \;:\; 0 \leq m_i \leq n_i \text{ for } i = 1, \ldots, d \right\}.$$

A progression is called proper if for some such $p_0, \overline{p}, \overline{n}$ it holds that $|P| = \prod_{1\leq i\leq d}(n_i + 1)$.

Freiman's Theorem [3] states that every set A with $|A + A| \leq c|A|$ is contained in a proper multidimensional progression P which is not much larger than A.

▶ **Theorem 7** (Freiman). *Let $(G, +)$ be a torsion free Abelian group. For every constant $c \in \mathbb{N}$, there exist $d, k \in \mathbb{N}$ such that every finite subset $A \subseteq G$ with $|A + A| \leq c|A|$ is contained in some proper d-dimensional arithmetic progression P of size at most $k|A|$.*

For proofs we refer to [12] or to the very nice lecture notes [4]. In our analysis of ω-automatic fields, we will need a consequence of Freiman's Theorem which ensures that a fairly large portion of such a set A is contained in an (one-dimensional) arithmetic progression.

▶ **Lemma 8.** *Let $(G, +)$ be a torsion free Abelian group. For every constant c there exist d, k such that for every finite subset $A \subseteq G$ with $|A + A| \leq c|A|$ there exists an arithmetic progression P with $|P \cap A| \geq \sqrt[d]{|A|}/k$.*

Proof. Since $|A + A| \leq c \cdot |A|$, by Freiman's Theorem there is a proper progression $Q = \left\{q_0 + \sum_{i=1}^{d} m_i q_i \,:\, 0 \leq m_i \leq n_i \text{ for } i = 1, \ldots, d\right\}$ such that $A \subseteq Q$ and $|Q| \leq k \cdot |A|$. We may assume that $n_d \geq n_i$ for all $i < d$, which implies $n_d + 1 \geq \sqrt[d]{|A|}$.

We will now take a closer look at the representation of A in Q. For every tuple $\overline{m} = (m_1, \ldots, m_{d-1}) \leq (n_1, \ldots, n_{d-1})$ we write $A_{\overline{m}}$ for the set

$$A_{\overline{m}} := \left\{ q_0 + \sum_{i=1}^{d-1} m_i q_i + m q_d \,:\, m \leq n_d \right\} \cap A.$$

Clearly, every set $A_{\overline{m}}$ is contained in the induced one dimensional arithmetic progression $P_{\overline{m}} = \{p_0 + np_1 \,:\, n \leq n_d\}$ with $p_0 = q_0 + \sum_{i=1}^{d-1} m_i q_i$ and $p_1 = q_d$.

All we need to show now is that there is an $\overline{m}$ with $|A_{\overline{m}}| \geq \sqrt[d]{|A|}/k$. Since Q is proper, it holds that $|Q| = \prod_{i=1}^{d}(n_i + 1)$. The sets $A_{\overline{m}}$ with $m_i \leq n_i$ form a partition of A into $\prod_{i=1}^{d-1}(n_i + 1)$ sets. It follows that there is a tuple $\overline{m}$ with

$$|A_{\overline{m}}| \geq \frac{|A|}{\prod_{i=1}^{d-1}(n_i + 1)} \geq \frac{|Q|/k}{\prod_{i=1}^{d-1}(n_i + 1)} = \frac{\prod_{i=1}^{d}(n_i + 1)}{k \prod_{i=1}^{d-1}(n_i + 1)} = \frac{n_d + 1}{k} \geq \frac{\sqrt[d]{|A|}}{k}.$$

◀

4 End-Equivalence

Two words $v, w \in \Sigma^\omega$ are end-equivalent, in short $v \sim_e w$, if they are equal from some position onwards, i.e. $v[n, \omega) = w[n, \omega)$ for some $n \in \mathbb{N}$. Making explicit a position m after which the words are equal, we obtain refined relations $\sim_e^m$; two words are $\sim_e^m$-equivalent (m-end-equivalent) if $v[m, \omega) = w[m, \omega)$. Clearly $v \sim_e w$ if, and only if, $v \sim_e^m w$ for some m and the $\sim_e^m$-classes partition any language into finite classes of size at most $|\Sigma|^m$.

End-equivalence plays an important role in the study of ω-regular languages. We first observe that every infinite regular language has an infinite $\sim_e$-class.

▶ **Lemma 9.** *Let L be an infinite ω-regular language. Then there is an infinite $\sim_e$-class $X \in L/\sim_e$.*

Proof. Since L is an ω-regular language, by [2] it has the form $L = \bigcup_{1 \leq i \leq n} U_i V_i^\omega$ for some (finite-word) regular languages U_i, V_i which are not empty. We consider two cases.

Suppose that $V_i^\omega = \{v_i^\omega\}$ for each i, in which case L is countable. Since L is infinite there is an i such that $U_i v_i^\omega$ is also infinite.

Observe that the words in $U_i v_i^\omega$ need not be pairwise $\sim_e$-equivalent. For example consider $U = 1^*, v = 01$. Here it holds that $(01)^\omega \in Uv^\omega$ and $(10)^\omega \in Uv^\omega$ but $(01)^\omega \not\sim_e (10)^\omega$. Nevertheless, all $w \in U_i v_i^\omega$ fall into one of at most $|v_i|$ many $\sim_e$-classes and therefore $U_i v_i^\omega / \sim_e$ must contain an infinite class.

In the other case, there is a V_i which contains two words w, v such that $w^\omega \neq v^\omega$. Set $U'_i := U_i w^*$, then $U'_i v^\omega \subseteq (U_i V_i^*) V_i^\omega = U_i V_i^\omega$. The language $U'_i v^\omega$ is infinite. Otherwise the language $w^* v^\omega$ would also be finite and therefore $w^i v^\omega = w^j v^\omega$ for some $i \neq j$. But then $w^l = v^k$ for some $k, l \in \mathbb{N}$ and therefore $w^\omega = v^\omega$, contradiction. Since $U'_i v^\omega$ is infinite, we know from the first part of the proof that $U'_i v^\omega / \sim_e$ contains an infinite $\sim_e$-class. ◀

In the following we examine which elements of a structure can be encoded in the same end-class. For this undertaking, it is handy to consider elements of the original structure $\sim_e$-equivalent with respect to some given presentation.

▶ **Definition 10.** Let $\mathfrak{A}$ be a structure with ω-automatic presentation $(\mathcal{L}, \pi)$ and $\sim$ an equivalence relation on the domain of the presentation. We say $B \subseteq A$ is $(\sim, \mathcal{L}, \pi)$-equivalent or $\sim$-equivalent in $(\mathcal{L}, \pi)$ iff $B \subseteq \pi(X)$ for some $X \in L/\sim$.

If the presentation that is considered is clear from the context we will often just write that the set is $\sim$-equivalent without mentioning the presentation explicitly.

Observe that an equivalence relation $\sim$ on L does not need to induce an equivalence relation on A. Indeed, an element of A can have several encodings in $\mathcal{L}$ and can thus occur in the image of more than one $\sim$-class. However, if a set B is $\sim$-equivalent in $(\mathcal{L}, \pi)$, then there are encodings of the elements of B such that these codings are pairwise $\sim$-equivalent.

As a first application, we use Lemma 9 to make a quick observation about injective ω-automatic presentations.

▶ **Lemma 11.** *Let $\mathfrak{A}$ be an infinite structure. Then, for every injective ω-automatic presentation $(\mathcal{L}, \pi)$ of $\mathfrak{A}$, there is an infinite set $M \subseteq A$ that is $\sim_e$-equivalent in $(\mathcal{L}, \pi)$.*

Proof. Since A is infinite, L must also be infinite and therefore by Lemma 9 there must be an infinite class $X \in L/\sim_e$. Since π is injective, it follows that $\pi(X)$ is an infinite set that is $\sim_e$-equivalent in $(\mathcal{L}, \pi)$. ◀

The following lemma gives us a useful property of sets that are $\sim_e^m$-equivalent in an ω-automatic presentation. Intuitively, it states that the image of an $\sim_e^m$-equivalent set B under a definable k-ary function can always be partitioned into a constant number of sets which are also $\sim_e^m$-equivalent and this constant only depends on the presentation but not on B.

▶ **Lemma 12.** *Let $\mathfrak{A}$ be a structure with ω-automatic presentation $(\mathcal{L}, \pi)$ and let $f : A^{k+\ell} \to A$ be a function that is* FO*-definable in $\mathfrak{A}$. Then there is a constant q such that, for every $m \in \mathbb{N}$, every $(\sim_e^m, \mathcal{L}, \pi)$-equivalent subset $B \subseteq A$ and every tuple $\overline{a} \in A^\ell$, the image $f(B^k, \overline{a})$ admits a partition into q $(\sim_e^m, \mathcal{L}, \pi)$-equivalent sets $C_0, \ldots, C_{q-1}$.*

Proof. Let $\mathcal{A}$ be a Büchi automaton with states $Q = \{0,...,q-1\}$ that recognises f in $(\mathcal{L}, \pi)$.

First, choose a tuple of words $v_{\overline{a}} \in \pi^{-1}(\overline{a})$. Since B is $\sim_e^m$-equivalent in $(\mathcal{L}, \pi)$, there is a set $M \subseteq L$ of pairwise $\sim_e^m$-equivalent words such that $\pi \restriction M$ is a bijection between M and B. For every tuple $\overline{b} \in B^k$ we denote by $v_{\overline{b}}$ the unique tuple in M^k with $\pi(\overline{v}_{\overline{b}}) = \overline{b}$. Now choose, for every tuple $\overline{b} \in B^k$, a word $f_{\overline{b}} \in \pi^{-1}(f(\overline{b}, \overline{a}))$. This means that the word $\langle v_{\overline{b}}, v_{\overline{a}}, f_{\overline{b}} \rangle$ is accepted by $\mathcal{A}$ for every tuple $\overline{b} \in B^k$. Let $\rho_{\overline{b}}$ be an accepting run of $\mathcal{A}$ on $\langle v_{\overline{b}}, v_{\overline{a}}, f_{\overline{b}} \rangle$.

We obtain a partition of M^k into sets $M_0, \ldots, M_{q-1}$ by defining $M_i := \{v_{\overline{b}} : \rho_{\overline{b}}[m] = i\}$. For every nonempty M_i fix a tuple $v_{\overline{b}} \in M_i$. We will now show that, for any $\overline{d}$ with $v_{\overline{d}} \in M_i$, there is an encoding of $f(\overline{d}, \overline{a})$ that is $\sim_e^m$-equivalent to $f_{\overline{b}}$.

The intuition is that we can simply replace the tail of $\langle v_{\overline{d}}, v_{\overline{a}}, f_{\overline{d}} \rangle$ by the tail of $\langle v_{\overline{b}}, v_{\overline{a}}, f_{\overline{b}} \rangle$ and obtain a new word that is accepted by $\mathcal{A}$. This will give us a new encoding of $f(\overline{d}, \overline{a})$

that is $\sim_e^m$-equivalent to $f_{\overline{b}}$. More formally, for every such $\overline{d}$ it holds that $\rho_{\overline{d}}[0,m)\rho_{\overline{b}}[m,\omega)$ is an accepting run on

$$\begin{aligned}\langle v_{\overline{d}}, v_{\overline{a}}, f_{\overline{d}}\rangle[0,m)\langle v_{\overline{b}}, v_{\overline{a}}, f_{\overline{b}}\rangle[m,\omega) &= \langle v_{\overline{d}}, v_{\overline{a}}, f_{\overline{d}}\rangle[0,m)\langle v_{\overline{d}}, v_{\overline{a}}, f_{\overline{b}}\rangle[m,\omega)\\ &= \langle v_{\overline{d}}, v_{\overline{a}}, f_{\overline{d}}[0,m)f_{\overline{b}}[m,\omega)\rangle.\end{aligned}$$

This holds since $\langle v_{\overline{d}}\rangle \sim_e^m \langle v_{\overline{b}}\rangle$ and $\rho_{\overline{d}}[m] = \rho_{\overline{b}}[m] = i$. But since $\mathcal{A}$ recognises f in $\mathcal{L},\pi$, it follows that $\pi(f_{\overline{d}}[0,m)f_{\overline{b}}[m,\omega)) = f(\overline{d},\overline{a})$.

So there is a $\sim_e^m$-class such that, for every $v_{\overline{d}} \in M_i$, the function value $f(\overline{d},\overline{a})$ has an encoding in this class. This implies that all the sets C_i' defined by $C_i' := \{f(\overline{b},\overline{a}) : v_{\overline{b}} \in M_i\}$ are $\sim_e^m$-equivalent in $(\mathcal{L},\pi)$. Obviously it holds that $\bigcup_{0\le i\le q-1} C_i' = f(B^k,\overline{a})$. We can now simply define C_i to be the set $C_i' - \bigcup_{0\le j<i} C_j'$ and obtain a partition $C_0,\ldots,C_{q-1}$ of $f(B^k,\overline{a})$ with the desired properties. ◀

From this result we infer that, in injectively ω-automatic structures, no FO-definable k-ary function can guarantee that the image of a finite set is always much larger than the set itself. We can make this precise by the notion of the minimal image size.

▶ **Definition 13.** The *minimal image size* of a function $f : A^k \to A$ over an infinite set A, $MIS_f : \mathbb{N} \to \mathbb{N}$, is given by $MIS_f(n) = \min\{|f(X^k)| : X \subseteq A, |X| = n\}$.

We now show that, for injectively presentable structures, the minimal image size of every FO-definable function grows at most linearly with n.

▶ **Lemma 14.** *Let $\mathfrak{A}$ be an infinite structure with an injective ω-automatic presentation. Then, for every* FO*-definable function f, it holds that $MIS_f(n) = \mathcal{O}(n)$.*

Proof. Suppose there is an injective automatic presentation $(\mathcal{L},\pi)$ of an infinite structure with FOdefinable function $f : A^k \to A$ such that MIS_f grows in a superlinear way.

Let $\mathcal{A} = (Q,\Sigma^{k+1},q_0,\Delta,F)$ be a Büchi automaton that recognizes f in $(\mathcal{L},\pi)$ and c the constant from Lemma 12 with respect to f and $(\mathcal{L},\pi)$. Now choose n such that $MIS_f(n) > c|\Sigma|\cdot n$. This is possible since MIS_f grows in a superlinear way. By Lemma 11 there is an infinite set $M \subseteq A$ that is $\sim_e$-equivalent in $(\mathcal{L},\pi)$. Therefore we can choose m such that there is a $(\sim_e^m,\mathcal{L},\pi)$-equivalent set of size at least n. Choose m minimal and let N be a $\sim_e^m$-equivalent set of maximal size. Then $n \le |N| \le |\Sigma|n$, otherwise we could have chosen m smaller.

Observe that if $MIS_f(n) = a$ then for all sets X with $|X| \ge n$ it holds that $|f(X)| \ge a$. By Lemma 12, $f(N^k)$ can be partitioned into c $\sim_e^m$-equivalent sets. One of these sets has size at least $|f(N^k)|/c > |\Sigma|n \ge |N|$. But this contradicts the maximality of N among all $\sim_e^m$-equivalent sets. ◀

Since pairing functions grow in a quadratic way, we obtain the following corollary.

▶ **Corollary 15.** *No infinite structure with an* FO*-definable pairing function admits an injective ω-automatic presentation.*

5 Fields of Characteristic 0

Every field of characteristic 0 contains, in a canonic way, a copy of $(\mathbb{N},+,\cdot)$. We first show that, whenever a field of characteristic 0 admits an ω-automatic presentation, the size of every $\sim_e$-equivalent set of natural numbers is bounded by a constant.

▶ **Lemma 16.** *Let $(K, +, \cdot)$ be a field of characteristic 0 with an ω-automatic presentation $(\mathcal{K}, \pi)$. Then there exists a constant q such that no $(\sim_e, \mathcal{K}, \pi)$-equivalent set N of natural numbers has size $|N| > q$.*

Proof. Consider the function $p : K \times K \to K$ defined by

$$p(x, y) := \frac{(x + y)(x + y + 1)}{2} + y.$$

Note that p restricted to the natural numbers is the Cantor pairing function. Now let q be the constant from Lemma 12 associated with the function p with respect to $(\mathcal{K}, \pi)$. Towards a contradiction, assume there exist sets of natural numbers that are $\sim_e$-equivalent in $(\mathcal{K}, \pi)$ and have more than q elements. This implies that, for some m, there exists a $(\sim_e^m, \mathcal{K}, \pi)$-equivalent set of natural numbers of size larger than q.

Fix such an m and let N be one of the largest such sets, i.e. such that $|N| \geq |M|$ for every $M \subseteq \mathbb{N}$ that is $\sim_e^m$-equivalent in $(\mathcal{K}, \pi)$. Now apply Lemma 12 and obtain a partition $M_1, \ldots, M_q$ of $p(N^2) \subseteq \mathbb{N}$ such that every M_i is $\sim_e^m$-equivalent in $(\mathcal{K}, \pi)$. Since p is a pairing function on $\mathbb{N}$, we have

$$|p(N^2)| = |N|^2 > q \cdot |N|.$$

It follows, by the pigeonhole principle, that there must be a M_i of size at least $|N|^2/q > |N|$. But, since M_i is a set of natural numbers that is $(\sim_e^m, \mathcal{K}, \pi)$-equivalent, this contradicts the choice of N as one of the largest sets. ◀

We will now generalise this statement from sets of natural numbers to arbitrary subsets of the field. The key here is the use Freiman's Theorem and Lemma 8 derived from it.

▶ **Lemma 3.** Whenever a field $(K, +, \cdot)$ of characteristic 0 admits an ω-automatic presentation $(\mathcal{K}, \pi)$ then the size of all $\sim_e$-equivalent subsets of $\mathcal{K}$ is bounded by a constant r.

Proof. We show that every set $M \subseteq K$ that is $\sim_e$-equivalent in $\mathcal{K}, \pi$ has size $|M| \leq r$.

Let q_1 be the constant q from Lemma 12 with respect to the function $(x - y_1)/y_2$, let q_2 be the constant q from Lemma 12 with respect to $+$, and let q_3 be the constant q from Lemma 16. We then set $c := \max\{q_1, q_2, q_3\}$ and let k, d be the constants from Freiman's Theorem with respect to c. Finally, we choose $r = c^{2d} \cdot k^d$.

Suppose there is a set M that is $\sim_e$-equivalent in $(\mathcal{K}, \pi)$ with $|M| > r$. Then we can also choose an m such that there are $\sim_e^m$-equivalent sets of size larger than r. Choose such a $\sim_e^m$-equivalent set M of maximal size.

It is easy to see that $|M + M| \leq c|M|$. Otherwise we could argue analogously to the proof of Lemma 16 and obtain a set C that is larger than M and also $\sim_e^m$-equivalent in $(\mathcal{K}, \pi)$.

By Lemma 8 there is an arithmetic progression $P = \{p_0 + mp_1 \,:\, m < n, m \in \mathbb{N}\}$ such that $N := M \cap P$ has size at least $\frac{\sqrt[d]{|M|}}{k}$. Now, by Lemma 12, we can partition the set $\{(a - p_0)/p_1 : a \in N\} \subseteq \mathbb{N}$ into at most c sets and obtain a $\sim_e^m$-equivalent set of natural numbers of size at least

$$\frac{\sqrt[d]{|M|}/k}{c} > \frac{\sqrt[d]{c^{2d}k^d}}{ck} = c.$$

But this contradicts Lemma 16. ◀

For injective presentations, Lemma 11 ensures infinite $\sim_e$-equivalent sets, but Lemma 3 forbids such sets, and therefore we directly obtain Theorem 1. For Theorem 2, more work is necessary. In the next section we show a result that will enable us to lift the above argument from injective presentations to general ones for fields with a definable linear order.

6 ω-Automatic Linear Orders

For uncountable fields of characteristic 0 with a definable linear order, we transfer the result of the previous section from injective presentations to general ω-automatic ones. The problem that we face if we consider non-injective presentations $(\mathcal{L}, \pi)$ is that we cannot simply infer that there are infinite $\sim_e$-equivalent sets. For example, $\sim_e$ is an ω-automatic equivalence relation and identifies all ultimately equal words.

But, as we will see, this cannot happen for ω-automatic presentations of uncountable linear orders. To prove this, it suffices to show that every ω-automatic presentation of an uncountable linear order $(A, <)$ contains an infinite regular subset such that the presentation, restricted to this subset, is an injective presentation of a suborder of $(A, <)$. In [9] Kuske had already shown that $(\{0,1\}^\omega, <_{lex})$ is embeddable into any ω-automatic uncountable linear order. He constructs, from a given ω-automatic presentation of such an order, a subpresentation that is a presentation of $(\{0,1\}^\omega, <_{lex})$. This subpresentation, however, is not ω-automatic: its domain is the complement of a language $\bigcup_{i \leq n} V_i U_i^\omega$ where the V_i are context free and and the U_i are regular. In particular, his presentations do not contain two $\sim_e$-equivalent words.

We will therefore present here a strengthening of Kuske's result. We show that every automatic presentation of an uncountable linear order contains an injective automatic presentation of $(\{0,1\}^\omega, <_{lex})$.

The main techniques originate from [6]. For a given ω-automatic presentation $(L, \approx, <)$ we construct finite words u, v_0, v_1 such that $u\{v_0, v_1\}^\omega \subseteq L$ and, for any two words $\alpha, \beta \in \{0,1\}^\omega$, it holds that

$$uv_{\alpha[0]}v_{\alpha[1]}v_{\alpha[2]}\ldots < uv_{\beta[0]}v_{\beta[1]}v_{\beta[2]}\ldots \text{ if, and only if, } \alpha <_{lex} \beta.$$

This means v_0 encodes 0, v_1 encodes 1 and the language $u\{v_0, v_1\}^\omega$ encodes $\{0,1\}^\omega$ in a natural way.

To handle these constructions, it is convenient to use an algebraic characterisation of ω-regular languages. More precisely, we use that a language L is ω-automatic if, and only if, there is an ω-semigroup morphism $g : \Sigma^\infty \to S$ from the free ω-semigroup $\Sigma^\infty = (\Sigma^*, \Sigma^\omega)$ into some finite ω-semigroup $S = (S_f, S_\omega)$ with $g^{-1}(g(L)) = L$. A broader introduction to the topic can be found in [10]. The advantage of having such a morphism g is that we know, whenever $g(u) = g(v)$ for two words u and v, that $u \in L$ if, and only if, $v \in L$. With this property it is much easier to ensure that the elements we construct have the properties that we want than by cutting apart and rearranging runs of automata.

▶ **Theorem 17.** *For every ω-automatic presentation $(L, \approx, <)$ of an uncountable linear order there exists a subset L' of L such that $(L', <\cap(L')^2)$ is an injective ω-automatic presentation of $(\{0,1\}^\omega, <_{lex})$.*

Proof. Let $\mathcal{L} = (L, \approx, <)$ be an ω-automatic presentation of an uncountable linear order. Since $\mathcal{L}$ is automatic, for each $\delta \in \{L, \approx, <\}$ there are ω-semigroup morphisms to finite ω-semigroups $S_\delta = (S_f^\delta, S_\omega^\delta)$:

$$()^L \;:\; \Sigma^\infty \to S_L, \quad ()^\approx \;:\; (\Sigma \times \Sigma)^\infty \to S_\approx, \quad ()^< \;:\; (\Sigma \times \Sigma)^\infty \to S_<$$

that recognise the corresponding relations. We set $F_L := (L)^L$ and for $\sim \in \{\approx, <\}$ we define $F_\sim := (\{\langle u, v\rangle \;:\; u \sim v\})^\sim$. We define c to be the size of the largest ω-semigroup among $S_L, S_\approx, S_<$, set $C := c^3$ and k as the least common multiple of the exponents of these semigroups.

As mentioned before, our goal is to show that there are $u, v_0, v_1 \in \Sigma^+$ with $v_0 \neq v_1$, $|v_0| = |v_1|$ such that $u\{v_0, v_1\}^\omega \subseteq L$ and for all $\alpha, \beta \in \{0,1\}^\omega$ it holds that $uv_\alpha < uv_\beta$ iff $\alpha <_{lex} \beta$ (here v_α stands for $v_{\alpha[0]}v_{\alpha[1]}v_{\alpha[2]}\ldots$). This obviously means, that $\mathcal{L}$ restricted to $u\{v_0, v_1\}^\omega$ is an injective ω-automatic presentation of $(\{0,1\}^\omega, <_{lex})$.

Before we start we need to fix some notions. We call an infinite set $H = \{h_0 < h_1 < h_2 < \ldots\} \subseteq \mathbb{N}$ a factorisation. For a given ω-semigroup morphism $g : \Sigma^\infty \to S$ and $w \in \Sigma^\omega$, we say that H is a g, e-homogeneous factorisation of w if $g(w[h_i, h_{i+1})) = e$ for all $i \in \mathbb{N}$.

The following lemma is basically a reformulation of Ramsey's theorem for countably infinite graphs to the language of ω-semigroup morphisms. It ensures the existence of factorisations that are homogeneous for several words and morphisms.

▶ **Lemma 18.** *For $i \in \{1, \ldots, n\}$ let $h_i : \Sigma_i^\infty \to S_i$ be a morphism into a finite ω-semigroup S_i and w_i a word over the corresponding alphabet. There exists a $G = \{g_1 < g_2 < g_3 \ldots\} \subseteq \mathbb{N}$ such that, for every i, G is an h_i,e_i-homogeneous factorisation of w_i for some idempotent semigroup element $e_i \in S_i$.*

Proof. This lemma is more or less a direct consequence of Ramsey's theorem: We colour every $\{k, l\} \subset \mathbb{N}$, $k < l$ with the tuple of semigroup elements $[h_i(w_i[k, l))]_{1 \leq i \leq n}$. From Ramsey's theorem we obtain a $G = \{g_1 < g_2 < g_3 \ldots\}$ such that all $\{g_i, g_j\}$, $i \neq j$ have the same colour. Having set $e_i := h_i(w_i[g_1, g_2))$, G obviously is a h_i,e_i-homogeneous factorisation of w_i and e_i is idempotent since $e_ie_i = h_i(w_i[g_1, g_2))h_i(w_i[g_2, g_3)) = h_i(w_i[g_1, g_3)) = e_i$. ◀

Now we are ready to start the construction. As a first step, we show that there are two words x_0 and x_1 and a factorisation H such that, regarding H, x_0 and x_1 are mapped to the same elements under every mentioned morphism. Later on we will use the obtained words to cut out suitable candidates for u, v_0 and v_1.

▶ **Lemma 19.** *There exist $x_0, x_1 \in L$ with $[x_i]_{\sim_e} \cap [x_{1-i}]_\approx = \emptyset$ and a factorisation $H = \{h_1 < h_2 < h_3 < \ldots\}$ such that, for some idempotent $e_L \in S_L$, H is a $()^L, e_L$-homogeneous factorisation of x_0 and x_1. For $\delta \in \{\approx, <\}$ there are idempotent e_δ, e_δ^{01} and e_δ^{10} such that*

- *H is a $()^\delta, e_\delta$-homogeneous factorisation of $\langle x_0, x_0\rangle$ and $\langle x_1, x_1\rangle$ and*
- *H is a $()^\delta, e_\delta^{01}$-homogeneous factorisation of $\langle x_0, x_1\rangle$ and*
- *H is a $()^\delta, e_\delta^{10}$-homogeneous factorisation of $\langle x_1, x_0\rangle$.*

Proof Sketch. Since $\mathcal{L}$ is a presentation of an uncountable structure, there are elements $y_0, \ldots, y_C$ such that $[y_i]_{\sim_e} \cap [y_j]_\approx = \emptyset$ for $i \neq j$. With Lemma 18 we obtain an H that is a homogeneous factorisation of all y_i and respectively all $\langle y_i, y_j\rangle$ under all mentioned morphisms. Since we have more elements y_i than triples $(a, b, c) \in S_L \times S_\approx \times S_<$ there must be some $i \neq j$ such that H is a $()^L, e_L$-homogeneous factorisation of y_i and y_j and for $\delta \in \{\approx, <\}$ a $()^\delta, e_\delta$-homogeneous factorisation of $\langle y_i, y_i\rangle$ and $\langle y_j, y_j\rangle$. ◀

We may also assume that H is coarse enough that $x_0[h_l, h_{l+1}) \neq x_1[h_l, h_{l+1})$ for all $l \in \mathbb{N}$. We need to modify x_0, x_1 a bit to ensure all the properties required later.

▶ **Lemma 20.** *There exist $y_0, y_1 \in L_A$ with $y_0 \not\approx y_1$ and a factorisation $G = \{g_1 < g_2 < g_3 < \ldots\}$ with the following properties:*

- *$y_0[0, g_1) = y_1[0, g_1)$ and $y_0[g_i, g_{i+1}) \neq y_1[g_i, g_{i+1})$ for all $i \in \mathbb{N}$.*
- *for $\delta \in \{\approx, <\}$ there is an element $\rightarrow_\delta \in S_f^\delta$ and idempotents $\Box_\delta, \uparrow_\delta, \downarrow_\delta \in S_f^\delta$ such that*
  - *$\langle y_0, y_0\rangle[0, g_1)^\delta = \rightarrow_\delta$,*
  - *$\langle y_0, y_0\rangle[g_i, g_{i+1})^\delta = \langle y_1, y_1\rangle[g_i, g_{i+1})^\delta = \Box_\delta$,*

- $\langle y_0, y_1\rangle[g_i, g_{i+1})^\delta = \uparrow_\delta$,
- $\langle y_1, y_0\rangle[g_i, g_{i+1})^\delta = \downarrow_\delta$ *and*
- $\rightarrow_\delta, \uparrow_\delta$ *and* $\downarrow_\delta$ *absorb* $\Box_\delta$.

Proof Sktech. We first construct y_0 and y_1 and then show that these have the desired properties. We define y_0 as $y_0 := x_1[0, h_2)x_0[h_2, \omega)$ and y_1 by

$$y_1[0, h_2) := x_1[0, h_2) \text{ and}$$
$$y_1[h_{2l}, h_{2l+2}) := x_1[h_{2l}, h_{2l+1})x_0[h_{2l+1}, h_{2l+2}) \text{ for } l \geq 1.$$

We set $G = \{h_{2kl+2} : l \in \mathbb{N}\}$ (remember k is the least common multiple of the exponents of the involved semigroups). It holds that $\langle y_0, y_1\rangle^\approx = \langle y_1, x_1\rangle^\approx$. So if $y_0 \approx y_1$ then also $y_1 \approx x_1$ and therefore $y_0 \approx x_1$. But $y_0 \sim_e x_0$ in contradiction to $[x_0]_{\sim_e} \cap [x_1]_\approx = \emptyset$. That the other postulated properties hold can be established by straightforward calculations. ◀

Now that we have y_0 and y_1, we are ready to construct u, v_0 and v_1. We set

$$u := y_1[0, g_0), \qquad v_0 := y_0[g_0, g_1) \text{ and } \qquad v_1 := y_1[g_0, g_1).$$

From this definition we immediately get for $\delta \in \{\approx, <\}$:

$$\langle u, u\rangle^\delta = \rightarrow_\delta, \quad \langle v_0, v_0\rangle^\delta = \langle v_1, v_1\rangle^\delta = \Box_\delta, \quad \langle v_0, v_1\rangle^\delta = \uparrow_\delta \text{ and}, \quad \langle v_1, v_0\rangle^\delta = \downarrow_\delta .$$

In the following we will omit the subscripts and just write $\rightarrow, \uparrow, \downarrow$ and $\Box$ since it will be obvious from the context which δ is meant.

We will now show that $u\{v_0, v_1\}^\omega$ has all the properties that were announced at the beginning of the proof.

▶ **Lemma 21.** $u\{v_0, v_1\}^\omega \subseteq L$

Proof. Let α be any sequence from $\{0, 1\}^\omega$.

$$(uv_\alpha)^L = y_1[0, g_0)^L(y_{\alpha[i]}[g_0, g_1)^L)_{i\in\mathbb{N}} = x_1[0, g_0)^L(e_L)^\omega = (x_1)^L \in F_L$$

This means every uv_α is in L and therefore $u\{v_0, v_1\}^\omega \subseteq L$. ◀

Next we show that at least some words from $u\{v_0, v_1\}^\omega$ do encode distinct elements.

▶ **Lemma 22.** $\rightarrow (\uparrow\downarrow)^\omega \notin F_\approx$.

Proof. First we see that $\rightarrow\uparrow^\omega \notin F_\approx$ since $\langle y_0, y_1\rangle^\approx = \rightarrow\uparrow^\omega$ and $y_0 \not\approx y_1$. We will make use of the transitivity of $\approx$ to show that also $\rightarrow (\uparrow\downarrow)^\omega \notin F_\approx$. Suppose $\rightarrow (\uparrow\downarrow)^\omega \in F_\approx$, then consider the words $u(v_0v_1v_0)^\omega$,$u(v_1v_0v_1)^\omega$ and $u(v_1v_1v_0)^\omega$. We have

$\langle u(v_0v_1v_0)^\omega, u(v_1v_0v_1)^\omega\rangle^\approx = \rightarrow (\uparrow\downarrow\uparrow)^\omega = \rightarrow (\uparrow\downarrow)^\omega$ and $\langle u(v_1v_0v_1)^\omega, u(v_1v_1v_0)^\omega\rangle^\approx = \rightarrow (\Box \uparrow\downarrow)^\omega = \rightarrow (\uparrow\downarrow)^\omega$.

Hence $u(v_0v_1v_0)^\omega \approx u(v_1v_0v_1)^\omega$ and $u(v_1v_0v_1)^\omega \approx u(v_1v_1v_0)^\omega$ and so by transitivity $u(v_0v_1v_0)^\omega \approx u(v_1v_1v_0)^\omega$, but $\langle u(v_0v_1v_0)^\omega, u(v_1v_1v_0)^\omega\rangle^\approx = \rightarrow (\uparrow \Box\Box)^\omega = \rightarrow\uparrow^\omega \notin F_\approx$, contradiction. ◀

We conclude our proof by showing that $<$ is indeed the lexicographic order on $u\{v_0, v_1\}^\omega$.

▶ **Lemma 23.** *Either it holds for every* $\alpha \neq \beta \in \{0, 1\}^\omega$ $uv_\alpha < uv_\beta$ *iff* $\alpha <_{lex} \beta$ *or it holds for every* $\alpha \neq \beta \in \{0, 1\}^\omega$ $uv_\alpha < uv_\beta$ *iff* $\beta <_{lex} \alpha$.

Proof Sketch. First observe that $u(v_0v_1)^\omega \not\approx u(v_1v_0)^\omega$ since $\langle u(v_0v_1)^\omega, u(v_1v_0)^\omega\rangle^{\approx} = \rightarrow (\uparrow\downarrow)^\omega \notin F_{\approx}$. Therefore either $u(v_0v_1)^\omega < u(v_1v_0)^\omega$ or $u(v_1v_0)^\omega < u(v_0v_1)^\omega$ holds. In the first case, we can show, for every $\alpha \neq \beta \in \{0,1\}^\omega$, that $uv_\alpha < uv_\beta$ iff $\alpha <_{lex} \beta$. The other case leads to the other part of the lemma's statement and can be shown analogously.

Using the idempotence and absorption properties of $\uparrow, \downarrow$ and $\rightarrow$ it is possible to write all elements $\langle uv_\alpha, uv_\beta\rangle^<$ as infinite products with no consecutive occurrences of two $\uparrow, \downarrow$ or $\square$ (except for infinite occurrences at the end). For every such infinite product ρ that originates from the image of a word $\langle uv_\alpha, uv_\beta\rangle$ with $\alpha <_{lex} \beta$, it is possible to find words $w_1, w_2, w_3 \in u\{v_0, v_1\}^\omega$ with $\langle w_1, w_2\rangle^< = \langle w_2, w_3\rangle^< = \rightarrow (\uparrow\downarrow)^\omega$ and $\langle w_1, w_3\rangle^< = \rho$. But, by transitivity of $<$, this implies $\rho \in F_<$ and therefore we get that $u\{v_0, v_1\}^\omega$ is ordered as desired. ◀

Taking all together, we get that $\mathcal{L}$ restricted to $u\{v_0, v_1\}^\omega$ is an injective ω-automatic presentation of $(\{0,1\}^\omega, <_{lex})$, which completes the proof of Theorem 17. ◀

Combining the above Theorem 17 and Lemma 3 proves Theorem 2 for uncountable fields of characteristic 0 with definable linear orders. Since countable ω-automatic structures have injective presentations [6], these are covered by Theorem 1, and Theorem 2 follows.

References

1 A. Blumensath and E. Grädel. Automatic structures. In *Proc. of LICS '00*, pages 51–62, 2000.

2 J. R. Büchi. On decision method in restricted second order arithmetic. In *Proc. of the International Congress on Logic, Methodology and Philosophy of Science*, pages 1–11, 1962.

3 G. Freiman. Foundations of a structural theory of set addition (in russian). *Gos. Ped. Inst. Kazan*, 1966.

4 B. Green. Structure theory of set addition. Unpublished lecture notes, available from `http://www.dpmms.cam.ac.uk/~bjg23/notes.html`, 2002.

5 G. Hjorth, B. Khoussainov, A. Montalbán, and A. Nies. From automatic structures to Borel structures. In *Proc. of LICS'08*, pages 431–441. IEEE Computer Society, 2008.

6 Ł. Kaiser, S. Rubin, and V. Bárány. Cardinality and counting quantifiers on ω-automatic structures. In *Proc. of STACS'08*, volume 1 of *LIPIcs*, pages 385–396. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Germany, 2008.

7 Khoussainov and Nerode. Automatic presentations of structures. In *Proc. of LCC: International Workshop on Logic and Computational Complexity*, 1994.

8 B. Khoussainov, A. Nies, S. Rubin, and F. Stephan. Automatic structures: Richness and limitations. *Logical Methods in Computer Science*, 3(2), 2007.

9 D. Kuske. Is Ramsey's theorem omega-automatic? In *Proc. of STACS'10*, volume 5 of *LIPIcs*, pages 537–548. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2010.

10 D. Perrin and J.-E. Pin. Semigroups and automata on infinite words. In *Semigroups, Formal Languages and Groups*, NATO Advanced Study Institute, pages 49–72. Kluwer academic publishers, 1995.

11 M. Rabin. Decidability of second-order theories and automata on infinite trees. *Transactions of the AMS*, 141:1–35, 1969.

12 I. Ruzsa. Generalized arithmetical progressions and sumsets. *Acta Math. Hungar.*, 65:379–388, 1994.

13 A. Tarski. *A Decision Method for Elementary Algebra and Geometry.* University of California Press, 1951.

14 T. Tsankov. The additive group of the rationals does not have an automatic presentation. *Journal for Symbolic Logic (to appear)*, 2011.