Banach-Mazur Games with Simple Winning Strategies

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Abstract

We discuss several notions of 'simple' winning strategies for Banach-Mazur games on graphs, such as positional strategies, move-counting or length-counting strategies, and strategies with a memory based on finite appearance records (FAR). We investigate classes of Banach-Mazur games that are determined via these kinds of winning strategies.

Banach-Mazur games admit stronger determinacy results than classical graph games. For instance, all Banach-Mazur games with ω-regular winning conditions are positionally determined. Beyond the ω-regular winning conditions, we focus here on Muller conditions with infinitely many colours. We investigate the infinitary Muller conditions that guarantee positional determinacy for Banach-Mazur games. Further, we determine classes of such conditions that require infinite memory but guarantee determinacy via move-counting strategies, length-counting strategies, and FAR-strategies. We also discuss the relationships between these different notions of determinacy.

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1 Introduction

Banach-Mazur games, played on directed graphs, are path-forming games where two players interact, in an infinite strictly alternating sequence of moves, to produce an infinite path through the graph. The objectives of the players are given by properties of infinite paths. What distinguishes Banach-Mazur games from classical games on graphs is that in every move of a Banach-Mazur game, the player can prolong the finite path that results from the previous moves by another finite path of arbitrary length (whereas in classical graph games, the possible moves of a player at a given position are just the outgoing edges).

The possibility to choose, at each point, an arbitrary finite path rather than just an edge changes the mathematical properties of the games considerably. It turns out that Banach-Mazur games have stronger determinacy properties, that the class of winning conditions that guarantee determinacy via reasonably simple winning strategies is much larger, and that winning regions of Banach-Mazur games are often algorithmically easier to compute than for classical graph games. In computer science, Banach-Mazur games have found applications for planning in nondeterministic domains [17], for the characterization of fair behaviour in concurrent systems [19], and for the semantics of timed automata [1, 2].

It should be noted that Banach-Mazur games on graphs are a special case of games in a general topological setting. Banach-Mazur games have been extensively studied in descriptive set theory (see [10, Chapter 6] or [11, Chapter 8.H]) and topology (see e.g. [18]).

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In their original variant (see [12, pp. 113–117]), the winning condition is given by a set W of real numbers; in the first move, one of the players selects an interval d_1 on the real line, then her opponent chooses an interval $d_2 \subset d_1$, then the first player selects a further refinement $d_3 \subset d_2$ and so on. The first player wins if the intersection $\bigcap_{n\in\omega} d_n$ of all intervals contains a point of W, otherwise her opponent wins. Similar games can be played on any topological space. Let $\mathcal{V}$ be a family of subsets of a topological space X such that each $V \in \mathcal{V}$ contains a non-empty open subset of X, and each nonempty open subset of X contains an element $V \in \mathcal{V}$. In the Banach-Mazur game defined on $(X, \mathcal{V})$ with winning winning condition $W \subseteq X$, the players take turns to choose sets $V_0 \supset V_1 \supset V_2 \supset \dots$ in $\mathcal{V}$, and Player 0 wins the play if $\bigcap_{n<\omega} V_n \cap W \neq \emptyset$. We refer to [18] for a survey on topological games and their applications to set-theoretical topology.

The Banach-Mazur Theorem characterizes determinacy of Banach-Mazur games in terms of topological properties of the winning condition. For the original game on the real line, it had been stated as a conjecture by Mazur in the Scottish book [12, Problem 43] with an addendum by Banach, dated August 4, 1935, saying that "Mazur's conjecture is true". The Banach-Mazur Theorem was published for the first time by Mycielski, Świerczkowski, and Zieba [14], without proof; the first published proof is due to Oxtoby [15].

▶ **Theorem 1.1** (Banach-Mazur). *Let $\mathcal{G}$ be a Banach-Mazur game on a topological space X with winning condition $W \subseteq X$.*

(i) Player 1 has a winning strategy for $\mathcal{G} \iff W$ is meager.
(ii) Player 0 has a winning strategy for $\mathcal{G} \iff W$ is co-meager in some basic open set.

From Theorem 1.1 we easily get strong results on determinacy of Banach-Mazur games.

▶ **Corollary 1.2.** *Every Banach-Mazur game $\mathcal{G}$ where the winning condition W has the Baire property is determined.*

Recall that a set X in a topological space has the *Baire property* if its symmetric difference with some open set is meager. Since Borel sets have the Baire property (see e.g. [16, Chapter 4]), it follows that Banach-Mazur games are determined for Borel winning conditions. Standard winning conditions used in computer science applications (in particular the ω-regular winning conditions) are contained in very low levels of the Borel hierarchy.

Banach-Mazur games on graphs fit into this general topological setting because the set $\mathsf{Paths}(G, v)$ of infinite paths through a graph G from v is a topological space whose basic open sets are $\mathcal{O}(x)$, the set of infinite prolongations of some finite path $x \in \mathsf{FinPaths}(G, v)$. Thus, when a player prolongs the finite path x played so far to a new path xy, she reduces the set of possible outcomes of the play from $\mathcal{O}(x)$ to $\mathcal{O}(xy)$, and she wins an infinite play $x_0x_1\dots$ if, and only if the unique infinite path in $\bigcap_{n<\omega} \mathcal{O}(x_0 \dots x_{n-1})$ belongs to the set of winning paths. Let us now formally define Banach-Mazur games on graphs.

▶ **Definition 1.3.** A *Banach-Mazur game* is given by a directed graph $G = (V, E)$ without terminal vertices, a distinguished initial vertex v and a winning condition $\mathsf{Win} \subseteq \mathsf{Paths}(G, v)$.

The game starts in vertex v with a move of Player 0 and the players strictly alternate. In a move, after a sequence of moves $\pi_0 \cdot \pi_1 \cdots \pi_{i-1}$ forming a finite path has already been played, the respective Player ($i \mod 2$) prolongs the path with another finite path π_i. Thus, a play results in an infinite path α and is won by Player 0 if $\alpha \in \mathsf{Win}$. Otherwise, Player 1 wins.

Here we study refined questions that are specific for Banach-Mazur games on graphs and which are motivated by similar investigations in the setting of classical graph games. Since strategies are, in general, very complicated objects that are difficult to handle algorithmically, it is relevant to study which games are determined via simple winning strategies.

The best studied notions of simple strategies are *positional strategies* and *finite-memory strategies*. It is a well-known result, due to Büchi and Landweber [5], that every classical graph game with an ω-regular winning condition is determined via a finite-memory winning strategy. Further, parity games, which are of fundamental importance for the verification of reactive systems and the evaluation of fixed-point logics, are even positionally determined [6, 13, 20]. This means that from every position of a parity game, one of the two players has a winning strategy whose moves only depend on the current position, not on the history of the play. Notice that these strong determinacy results do only depend on the winning condition of the game, and not at all on the game graph. We therefore can say that ω-regular winning conditions *guarantee determinacy via finite-memory strategies* and parity winning conditions *guarantee positional determinacy* (for classical graph games).

It is not difficult to see that results of this kind carry over immediately to Banach-Mazur games. But actually one can prove much stronger determinacy results for Banach-Mazur games than for classical graph games. In particular, finite-memory strategies for Banach-Mazur games can always be translated into positional strategies [3, 7].

▶ **Theorem 1.4.** *A Banach-Mazur game that is determined via a finite-memory winning strategy is in fact positionally determined.*

▶ **Theorem 1.5.** *All ω-regular winning conditions guarantee positional determinacy for Banach-Mazur games.*

Thus, for Banach-Mazur games, positional determinacy is a much stronger notion than for classical graph games, whereas determinacy via finite-memory strategies is uninteresting. Now, a closer look at games that require infinite memory strategies reveals that the underlying infinite memory structures are often very simple. In many cases it suffices to store, on an infinite set of colours, say the maximal colour seen so far, or to have access to a counter for the number of moves, or to the length of the path constructed so far.

In this paper we shall consider several notions of simple strategies for Banach-Mazur games, and study the question which winning conditions guarantee determinacy for Banach-Mazur games via such strategies. Let us make this question more precise.

▶ **Definition 1.6.** Let $W \subseteq C^\omega$ be a set of infinite words on some alphabet C of colours. On every *arena* (G, v, Ω), given by a directed graph $G = (V, E)$ without terminal vertices, a distinguished initial position v and a function $\Omega : V \to C$, the set W defines the winning condition $\{\alpha \in \mathsf{Paths}(G, v) : \Omega(\alpha) \in W\}$.

For a class $\mathcal{A}$ of arenas, a class $\mathcal{S}$ of strategies and a class $\mathcal{W}$ of winning conditions, we say that $\mathcal{W}$ *guarantees determinacy via* $\mathcal{S}$ *on* $\mathcal{A}$, if all Banach-Mazur games on an arena $A \in \mathcal{A}$ with a winning condition $W \in \mathcal{W}$ are determined, and the winning player has a winning strategy in $\mathcal{S}$.

If we omit the class of arenas or strategies, we mean *on all arenas* or *via any strategy*.

Given that ω-regular winning conditions guarantee positional determinacy, we consider more general properties that depend on an infinite set of colours. Such conditions arise naturally in many contexts. In pushdown games for instance, stack height and stack contents are natural parameters that may take infinitely many values, in other scenarios Muller or parity conditions have been combined with requirements that an infinite portion of the game graph is visited. A natural class of conditions that go beyond ω-regularity are infinitary Muller conditions, based on a (countably) infinite set of colours. Our main results can be summarized as follows:

- We observe that every prefix-independent winning condition that guarantees positional determinacy on all strongly connected graphs and on all infinite acyclic ones in fact guarantees positional determinacy on all arenas.
- We introduce the notion of a *finitely based* Muller condition and show that it provides a necessary, and on strongly connected game graphs also sufficient, criterion for guaranteeing positional determinacy.
- We show that every Muller winning condition where the winning collection of one of the two players is countable guarantees determinacy via move-counting strategies.
  On the other side, move-counting strategies are of limited use for winning conditions that are not prefix-independent, whereas length-counting strategies, which can always simulate move-counting strategies, are sufficient for a larger class of winning conditions.
- Infinitary Muller winning conditions that guarantee determinacy via move-counting strategies do so also via strategies based on finite appearance records, called FAR strategies. However, such a result does not extend to more general winning conditions, not even for prefix-independent ones. We argue that the classes of games that guarantee determinacy via move-counting or length-counting strategies and via FAR strategies are in a sense incomparable.

2 Simple strategies for Banach-Mazur games

In this section we make precise the classes of simple strategies that we examine. We start by specifying what we mean by a memory strategy, and then formalize special cases of memory strategies. Furthermore, we shall discuss two classes of strategies that depend on a counter rather than on a memory structure, namely the move-counting strategies and the length-counting strategies.

▶ **Definition 2.1.** A *memory structure* $\mathfrak{M} = (M, \mathsf{init}, \mathsf{update})$ consists of a (possibly infinite) set M, called the *universe*, an *initialization function* $\mathsf{init}\colon V \to M$ and an *update function* $\mathsf{update}\colon M \times V \to M$.

A *memory strategy* f with memory $\mathfrak{M}$ is a strategy that can be characterized by a function $f'\colon V \times M \to \mathsf{FinPaths}(G)$ such that $f(\pi \cdot v) = f'(v, \mathsf{up}(\pi \cdot v))$, where up is inductively defined by $\mathsf{up}(v) := \mathsf{init}(v)$ and $\mathsf{up}(\pi \cdot v) := \mathsf{update}(\mathsf{up}(\pi), v)$.

Positional strategies. Strategies that do only depend on the current position and not at all on the history of a play are given by a function $f\colon V \to \mathsf{FinPaths}(G)$. They can be viewed as a trivial case of memory strategies, with a single memory location (i.e. $|M| = 1$). Positional strategies will be discussed in Section 3.

Beyond positional strategies, the most prominent classes of memory strategies are the *finite-memory strategies* which are very important for classical graph games. For Banach-Mazur games, however, they are not really interesting since by Theorem 1.4 they collapse to positional strategies. It is therefore relevant to consider simple classes of infinite memory structures.

FAR strategies. Finite appearance records have been introduced in [8] for studying Muller games with an infinite set of colours. Although, over an infinite set of colours, one can easily construct Muller games that do not admit finite-memory winning strategies, these games are often solvable by strategies with very simple infinite memory structures. In many cases, the required memory is essentially a finite collection of previously seen colours. This motivated the definition of a finite appearance record (FAR) which generalizes the latest appearance records (LAR) used for finitely coloured Muller games. In an FAR we store tuples of previously encountered colours or some other symbols from a finite set. Additionally the update function is restricted, so that new values of the memory can be equal only to the values stored before or to the currently seen colour.

▶ **Definition 2.2.** Let $(G, v_0, \Omega\colon V \to C)$ be an arena where positions are labeled with colours from an infinite set C. A *finite appearance record* (FAR) of dimension d for such an arena A is an infinite memory structure $\mathfrak{F}(A, d) = (M, \mathsf{init}, \mathsf{update})$ with universe $M := (C \cup \Sigma)^d$ for some finite alphabet Σ, with an initializing function $\mathsf{init}\colon V \to M$ and an update function $\mathsf{update}\colon M \times V \to M$ satisfying, for every memory state $m = (m_0, \cdots, m_{d-1})$ and every vertex v, that $\mathsf{update}(m, v) \in (\{m_0, \cdots, m_{d-1}\} \cup \{\Omega(v)\} \cup \Sigma)^d$.

An *FAR strategy* is a memory strategy that uses an FAR as memory structure. A discussion of FAR strategies will be given in Section 5.

For several interesting winning conditions that require infinite memory structures, it suffices to have access to a strictly increasing sequence of natural numbers. To provide such access in a systematic way without giving away too much information about the actual history, we can count the moves or the length of the path produced so far.

Move-counting strategies. A *move-counting strategy* is a strategy that can be characterized by a function $g\colon V \times \omega \to \mathsf{FinPaths}(G)$, such that the i-th move according to the strategy is given by $g(v, i)$ (for the respective v).

Accordingly, when playing a move-counting strategy g, the sequence of moves determined by the strategy are $g(\_, 0), g(\_, 1), g(\_, 2), \cdots$. Section 4 features a discussion of these strategies, proving their usefulness in many settings, but also showing their limitations.

Length-counting strategies. A *length-counting strategy* is a strategy f that can be characterized by a function $g\colon V \times \omega \to \mathsf{FinPaths}(G)$, such that $f(\pi \cdot v) = g(v, |\pi|)$. It turns out that these strategies are strictly stronger than move-counting strategies, thus the additional information about the history indeed contributes to the strength (cf. Section 4.3).

3 Positional Determinacy

This section is organized as follows. We begin with a general discussion and analysis of prefix-independent winning conditions. This is followed by a discussion of Muller winning conditions with infinitely many colours. We introduce the notion of a finitely based Muller condition and prove that it provides a necessary, and on strongly connected game graphs also sufficient, criterion for guaranteeing positional determinacy.

3.1 Prefix-independent winning conditions

▶ **Definition 3.1.** A winning condition $W \subseteq C^\omega$ is *prefix-independent* if for all $u, v \in C^*, x \in C^\omega$ it holds that $ux \in W \iff vx \in W$.

Obviously, if the winning condition of a Banach-Mazur game is prefix-independent, and u, v are two positions such that v is reachable from u, then

(a) if Player 0 wins (positionally) the game from initial position v, then she also does so from u, and
(b) if Player 1 wins (positionally) the game from initial position u, she also does so from v.

In particular, if a player wins from some position in a strongly connected component of the given game graph, then she wins from all positions in that component.

We first remark that prefix-independent winning conditions do not necessarily guarantee determinacy, let alone positional determinacy. A specific example of a nondetermined Banach-Mazur game with prefix-independent winning condition can be obtained on the basis of free ultrafilters, modifying a well-known construction of nondetermined Gale-Stewart games or Banach-Mazur games (see e.g. [4, 7]).

It turns out that to characterize prefix-independent winning conditions that guarantee positional determinacy, two specific classes of game graphs need to be studied, namely the strongly connected ones and the acyclic ones.

▶ **Proposition 3.2.** *Suppose that $W \subseteq C^\omega$ is prefix-independent.*

1. *If W guarantees positional determinacy on all strongly connected finite graphs, then it does so on all finite graphs.*
2. *If W guarantees positional determinacy on all strongly connected graphs and on all (non-terminating) acyclic ones, then it does so on all game graphs.*

Proof. Given a game graph G and an initial position v, we decompose G into its strongly connected components (SCC). A SCC is called terminal, if it does not have an outgoing edge. We distinguish three cases.

1. There is a terminal SCC of G, which is reachable from v and on which Player 0 wins (with a positional strategy). Then Player 0 also wins from v with a positional strategy.
2. For all nodes w reachable from v, there is a terminal SCC reachable from w on which Player 1 wins (with a positional strategy). Then she also wins from v with a positional strategy.

For finite graphs there are no other possibilities, which proves the first claim.

For infinite graphs, if neither (1) nor (2) holds, then Player 0, to avoid losing, must move in her first move to a position w from which no terminal SCC at all is reachable. The subgraph G_w of all points reachable from w need not be acyclic. We transform it into an acyclic graph $\tilde{G}_w$ as follows. In every strongly connected component H of G_w, we select a node $h \in H$ and unravel H to a tree $\mathcal{T}(H)$ with root h. We replace H by $\mathcal{T}(H)$. Further, for every edge (u, u') of G_w such that u and u' belong to different components H, H', we add edges from each copy of u in $\mathcal{T}(H)$ to all copies of u' in $\mathcal{T}(H')$. There is a canonical homomorphism $h\colon \tilde{G}_w \to G_w$. By assumption, one of the players has a positional winning strategy f for the game $(\tilde{G}_w, w, \mathsf{Win})$. Without loss of generality, we can assume f never stays in the same component. More precisely, from each node $y \in \mathcal{T}(H)$ the move $y \mapsto f(y)$ leaves the tree $\mathcal{T}(H)$. Indeed if this is not the case, then take any potential next move p of the opponent that takes the play from the end-point of $f(y)$ to a position $z \notin \mathcal{T}(H)$, and replace $f(y)$ by $f(y) \cdot p \cdot f(z)$. Any infinite path that is consistent with this modified strategy is also consistent with the original strategy, so the modified positional strategy is winning as well.

Further, the positional strategy f on $\tilde{G}_w$ translates into a positional strategy f' on G_w. For every infinite play that is consistent with f' in G_w there exists an infinite play in $\tilde{G}_w$ that is consistent with f and has the same labeling. Hence f' is also a winning strategy. ◀

The following examples show that it is indeed necessary to consider strongly connected game graphs and acyclic ones separately.

▶ **Example 3.3.** Given two winning conditions $W \subseteq C^\omega$ and $Z \subseteq D^\omega$, let $W + Z$ be the set of infinite words $(w, z) \in (C \times D)^\omega$ such that $w \in W$ *or* $z \in Z$. Clearly if W and Z are prefix-independent, then so is $W + Z$. Let now $\exists$-Inf(C) be the set of words $x \subseteq C^\omega$ such that some colour $c \in C$ occurs infinitely often in x.

Notice that every winning condition of the form $W + \exists$-Inf guarantees positional determinacy on strongly connected game graphs: Player 0 wins by always going back to the initial vertex. Let us now assume that the winning condition $W \subseteq C^\omega$ admits a nondetermined (or not positionally determined) game on some acyclic game graph $G = (V, E)$ with colouring $\Omega\colon V \to C$, so that we can add a second colouring $\Omega'\colon V \to D$ with the property that no infinite path has a colour $d \in D$ appearing infinitely often. Such a second colouring can always be added if the game graph is countable (in which case Ω' can be any injective function), and more generally in all cases where the reachability order has cofinality ω.

We thus obtain a game with winning condition $W + \exists$-Inf that is positionally determined on strongly connected graphs, but nondetermined (or not positionally determined) on acyclic graphs.

On acyclic graphs, let $v \prec w$ mean that w is reachable from v.

▶ **Example 3.4.** A winning condition that is positionally determined on countable acyclic graphs, but not on strongly connected ones is universal infinity $\forall$-Inf(C) (for infinite C), the set of infinite words $x \in C^\omega$ such that *all* $c \in C$ occur infinitely often in x.

Given a countable game graph G and a node v, let $R(v)$ be the set of all *colours* that are reachable from v. Clearly, if for all v with $v_0 \prec v$ there exists some w with $v \prec w$ and $R(w) \subsetneq C$, then Player 1 has a positional winning strategy for the game of universal infinity on G. If this is not the case, then Player 0 can move to a node v such that $R(w) = C$ for all nodes w reachable from v.

Let $C = \{c_0, c_1, \dots\}$. Choose an injective function ℓ that associates with every node w a natural number $\ell(w)$. Note that from every w there exists a reachable z such that $\ell(z') > \ell(w)$ for all z' with $z \prec z'$. Let $f(v_0)$ be the path from v_0 to v. For every node w reachable from v, let $f(w)$ be a finite path from w on which at least the colours $c_0, \dots c_{\ell(w)}$ occur and which ends in a vertex z with $\ell(z) > \ell(w)$ and $\ell(z') > \ell(w)$ for all z' with $z \prec z'$. This induces a positional winning strategy for Player 0.

On the other side, it is clear that Player 0 wins the game of universal infinity on the strongly connected graph on C, but not with a positional strategy (if C is infinite).

These findings can be summarized as follows.

▶ **Proposition 3.5.** *Each of the following three properties is satisfied by certain prefix-independent winning conditions W:*

1. *W guarantees positional determinacy on all strongly connected game graphs, but does not even guarantee determinacy on acyclic game graphs.*
2. *W guarantees determinacy on all game graphs and positional determinacy on strongly connected game graphs, but does not guarantee positional determinacy on acyclic game graphs.*
3. *W guarantees positional determinacy on countable acyclic game graphs, but not on strongly connected ones.*

3.2 Muller winning conditions

A Muller condition is any property of infinite sequences $x \subseteq C^\omega$ that depends only on which symbols $c \in C$ occur infinitely often in x. Muller conditions are of crucial importance in automata theory and in the theory of infinite games. It is one of the standard acceptance conditions for automata on infinite words or infinite trees.

▶ **Definition 3.6.** A Muller condition on a set C is written in the form $(\mathcal{F}_0, \mathcal{F}_1)$ where $\mathcal{F}_0 \subseteq \mathcal{P}(C)$ and $\mathcal{F}_1 = \mathcal{P}(C) \setminus \mathcal{F}_0$. Given a game graph $G = (V, E)$ whose nodes are labeled by a function $\Omega \colon V \to C$, a play $\alpha \in \mathsf{Paths}(G, v)$ is won by Player σ if, and only if, the set of colours occurring infinitely often on α belongs to $\mathcal{F}_\sigma$. We will always assume that C is countable.

For classical graph games, where in each move the players select an edge rather than a path, complete characterizations of the Muller conditions that guarantee positional determinacy are known. For the case where the set C of colours is finite, Zielonka [20] has shown that $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees positional determinacy for classical graph games if, and only if, neither $\mathcal{F}_0$ nor $\mathcal{F}_1$ contains a strong split, which means that there do not exist two sets $X, Y \in \mathcal{F}_\sigma$ such that $X \cap Y \neq \emptyset$ and $X \cup Y \in F_{1-\sigma}$. For the case of an arbitrary set of colours, Grädel and Walukiewicz have provided the following general characterization [9].

▶ **Theorem 3.7** ([9]). *For every Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ the following are equivalent.*

1. $(\mathcal{F}_0, \mathcal{F}_1)$ *guarantees positional determinacy for classical graph games.*
2. $\mathcal{F}_0$ *and* $\mathcal{F}_1$ *are closed under union of chains, non-empty intersections of chains, and have no strong splits.*
3. $(\mathcal{F}_0, \mathcal{F}_1)$ *is described by a Zielonka path of co-finite sets.*
4. $(\mathcal{F}_0, \mathcal{F}_1)$ *reduces to a parity condition on an ordinal* $\alpha \leq \omega$.

We would like to obtain a similar characterization for Banach-Mazur games with Muller conditions. Here, the case of a finite set of colours is trivial: *all* Muller conditions over a finite set guarantee positional determinacy. Indeed, Muller conditions over a finite set are ω-regular so this follows from Theorem 1.5. For a simple direct proof, see [3, 7].

Let us hence consider Muller conditions over an infinite set of colours. In that case Muller conditions need not even guarantee determinacy, and even if they do, they need not guarantee positional determinacy. This can be proved with a Muller condition over a non-principal ultrafilter, and the condition of universal infinity.

▶ **Definition 3.8.** $\mathcal{F} \subseteq \mathcal{P}(C)$ is *finitely based* if for each infinite $Z \in \mathcal{F}$, there is a finite basis $X \subseteq Z$ such that all Y with $X \subseteq Y \subseteq Z$ also belong to $\mathcal{F}$.

▶ **Lemma 3.9.** *For a Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ we have that if $\mathcal{F}_\sigma$ is finitely based then $\mathcal{F}_{1-\sigma}$ is closed under unions of chains.*

Proof. Assume that $\mathcal{F}_\sigma$ is finitely based, and let X be the union of an infinite chain $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ of sets $X_n \in \mathcal{F}_{1-\sigma}$. If it were the case that $X \in \mathcal{F}_\sigma$, then we could take a finite basis Z of X. Clearly $Z \subseteq X_n$ for some n, which would imply that $X_n \in \mathcal{F}_\sigma$. ◀

Note that the converse does not hold: the set $\mathcal{F}_1 := \{\omega \setminus \{i\} \mid i \in \omega\} \subseteq \mathcal{P}(\omega)$ is closed under unions of chains, as the only chains are the trivial chains $(X_j)_{j \in \omega}$ with $X_j = X_k$ for all j, k. But $\mathcal{F}_0 := \mathcal{P}(\omega) \setminus \mathcal{F}_1$ is not finitely based, as $\omega \in \mathcal{F}_0$, but for all finite sets $Z \subseteq \omega$ there exists some $Y = \omega \setminus \{i\} \in \mathcal{F}_1$ with $Z \subseteq Y \subseteq \omega$.

▶ **Proposition 3.10.** *A Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees positional determinacy for strongly connected Banach-Mazur games if, and only if, both $\mathcal{F}_0$ and $\mathcal{F}_1$ are finitely based.*

Proof. We have to show that for all strongly connected game graphs G with labeling $\Omega\colon V \to C$, the associated Banach-Mazur game with winning condition $(\mathcal{F}_0, \mathcal{F}_1)$ is positionally determined. Let X be a finite basis of $\Omega(V) \in \mathcal{F}_\sigma$. By selecting from each node v a finite path from v that goes through all colours in X we obtain a positional winning strategy for Player σ. Indeed, in any play that is consistent with f, the set Y of colours seen infinitely often satisfies $X \subseteq Y \subseteq \Omega(V)$ and thus belongs to $\mathcal{F}_\sigma$.

For the converse, suppose that $\mathcal{F}_\sigma$ is not finitely based. Then there exists an infinite $Z \in \mathcal{F}_\sigma$ such that for all finite subsets $X \subseteq Z$ there exists a set Y between X and Z that belongs to the opponent. On the complete directed graphs with vertex set Z, Player σ can obviously win from any initial position v by ensuring that all vertices are seen infinitely often. But she cannot win with a positional strategy, because in the unique move from v given by that strategy, only a finite set X of colours is seen. Hence, in order to win against a positional strategy the opponent just has to make sure that all colours from Y are seen infinitely often and that all her moves end at v. Hence $(\mathcal{F}_0, \mathcal{F}_1)$ does not guarantee positional determinacy on strongly connected game graphs. ◀

Open problem: *Do finitely based Muller conditions guarantee positional determinacy on all game graphs?*

To see that Muller conditions that are not finitely based do not guarantee positional determinacy on acyclic graphs, we consider games on the graph $G = (\omega_1, <, 0)$ where ω_1 denotes the first uncountable ordinal. Since the number of colours is countable, we can assume a labeling $\Omega\colon \omega_1 \to \omega$.

We first observe that in general one cannot guarantee positionally that infinitely many colours occur infinitely often.

▶ **Proposition 3.11.** *On $G = (\omega_1, <, 0)$ with an arbitrary colouring $\Omega\colon \omega_1 \to \omega$, none of the two players can guarantee with a positional strategy that the set of colours seen infinitely often is infinite.*

Let $\mathcal{F}_\sigma \subseteq \mathcal{P}(C)$ be not finitely based. This means that there exists some $X \in \mathcal{F}_\sigma$ such that for all finite $Y \subseteq X$ there is a set Z with $Y \subseteq Z \subseteq X$ and $Z \in \mathcal{F}_{1-\sigma}$. Take any colouring $\Omega\colon \omega_1 \to X$ such that $\Omega^{-1}(x)$ is unbounded for all $x \in X$ so that Player σ can enforce that all $x \in X$ are seen infinitely often. However, against any positional strategy of Player σ, the opponent can win.

4 Determinacy via move-counting strategies

It was shown in the previous section that infinitely coloured Muller winning conditions which are finitely based guarantee positional determinacy on strongly connected game graphs, but that this is no longer true for Muller conditions which are not finitely based. As a matter of fact, this can be seen even in simple examples with only one winning set of Player 0.

▶ **Example 4.1.** Consider the Banach-Mazur game on the completely connected graph with vertex set ω, where Player 0 wins if and only if every $n \in \omega$ is seen infinitely often (which can e.g. be formulated as a Muller winning condition with $\mathcal{F}_0 = \{\omega\}$).

Clearly, Player 0 can win this game, e.g. by seeing larger and larger initial subsets of the natural numbers in every move. However, she does not have a positional winning strategy, as for every vertex n, only finitely many numbers are seen in the unique move from n.

Still, the required information to win in the above example is simple. In fact, only a counter is required, as Player 0 can, e.g., win by seeing the first i numbers in the i-th move. This motivates move-counting strategies.

It turns out that a large class of infinitely coloured Muller winning conditions guarantees determinacy via move-counting strategies, namely the class of those conditions where one set of winning sets is at most countable. Before we establish this, we prove that the class of singleton Muller winning conditions guarantees determinacy via move-counting strategies.

Note that to show general determinacy of these classes of winning conditions, we place the conditions in the Borel hierarchy, which suffices because all Borel sets have the Baire property. Furthermore, to simplify arguments we assume that the set of colours equals ω.

4.1 Singleton Muller winning conditions

The first kind of Muller winning conditions we consider are those where either $\mathcal{F}_0$ or $\mathcal{F}_1$ is a singleton set, i.e. $\mathcal{F}_0 = \{F\}$ or $\mathcal{F}_1 = \{F\}$. All these winning conditions guarantee determinacy: let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a Muller winning condition such that $\mathcal{F}_0 = \{F\}$ for some $F \subseteq \omega$ (the case where $\mathcal{F}_1$ is a singleton is analogous). Let Y_k^n be the set of those finite paths starting in v_0 on which colour k is seen at least n times. Then the set of all infinite paths on which k is seen infinitely often can be written as $X_k := \bigcap_{n\in\omega} Y_k^n \cdot V^* \cap \mathsf{Paths}(G, v_0) \in \Pi_2^0$. The set of all winning paths of Player 0 can hence be written as $\mathsf{Win}_0 = X_F := \bigcap_{k\in F} X_k \cap \bigcap_{k\notin F} X_k^{\mathcal{C}}$, and as ω is countable, it follows that $\mathsf{Win}_0 \in \Pi_3^0$. Thus, $\mathcal{G}$ is determined.

▶ **Theorem 4.2.** *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a Muller winning condition such that either $\mathcal{F}_0$ or $\mathcal{F}_1$ is a singleton. Then $\mathcal{G}$ is determined via move-counting strategies.*

Proof. As the case where $\mathcal{F}_1$ is a singleton differs from the one where $\mathcal{F}_0$ is only in the importance of the opening move, we focus on the latter. Assume thus that $\mathcal{F}_0 = \{F\}$ for some $F \subseteq \omega$. As $\mathcal{G}$ is determined, it suffices to show that if Player σ has a winning strategy, then she also has a move-counting one, both for $\sigma = 0$ and $\sigma = 1$.

Assume first that Player 0 has a winning strategy f. Let v be the vertex in which the opening move $f(v_0)$ ends. We claim that for any v' reachable from v it holds that

(i) all of F can still be seen from v' on, and
(ii) every $b \notin F$ can be prevented from being seen infinitely often (i.e. a v'' is reachable from where on b cannot be seen again).

If (i) was false, Player 1 would trivially win from v' and hence from v. If (ii) was false, Player 1 would win from v' and also from v, as she can force some $b \notin F$ to be seen infinitely often. Both would contradict f being a winning strategy.

Let g be the move-counting strategy defined as follows:

- $g(v_0, 0) := f(v_0)$
- If v is reachable after $f(v_0)$, then $g(v, n) := \vartheta_0 \cdot \vartheta_1$, where ϑ_0 is a shortest path such that the first n colours from F are seen at least once, and ϑ_1 is a shortest path prolonging ϑ_0 such that the minimal $b \notin F$ that can still be seen infinitely often after ϑ_0 cannot be seen again. (This is always possible because of (i) and (ii).)
- For all other v, n, set $g(v, n)$ to some arbitrary, but fixed successor.

Let α be a play that is consistent with g. We need to show that α is won by Player 0. Because of the ϑ_0-parts of Player 0's moves, and because ω is countable, it follows that

$F \subseteq \mathsf{Inf}(\alpha)$. Because of the ϑ_1-parts, it follows that $\mathsf{Inf}(\alpha) \subseteq F$, thus α is indeed won by Player 0.

Assume now that Player 1 has a winning strategy. Then, for any vertex v reachable from v_0 it must hold that a vertex v' is reachable such that from v' on, F cannot be seen completely, or that a vertex v'' is reachable, from where on some $b \notin F$ can always be seen. (Otherwise, Player 0 would have a move-counting winning strategy by the above.) However, in any case Player 1 has a positional winning strategy - move somewhere from where on F cannot be seen completely, or enforce the minimal $b \notin F$ for which this is possible - and thus also a move-counting one. ◀

▶ **Corollary 4.3.** *Let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller winning condition over a countably infinite set of colours where either $\mathcal{F}_0$ or $\mathcal{F}_1$ is a singleton. Then $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees determinacy via move-counting strategies.*

4.2 Countably infinite Muller winning conditions

We now consider Muller winning conditions where either $\mathcal{F}_0$ or $\mathcal{F}_1$ is still countable. With the notation from above, in the case where $\mathcal{F}_0$ is countable, we can write Win_0 as $\mathsf{Win}_0 = \bigcup_{F \in \mathcal{F}_0} X_F \in \Sigma^0_4$. Again, winning conditions of this kind guarantee determinacy.

Because the proofs only differ in the treatment of the opening move, we focus on the case where $\mathcal{F}_0$ is countably infinite. We show that these winning conditions guarantee determinacy via move-counting strategies, but in contrast to the singleton case, both players need move-counting strategies, and it is no longer true that one of the players can already win positionally if she wins at all. Towards the theorem, we observe the following lemma:

▶ **Lemma 4.4.** *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a Muller winning condition where $\mathcal{F}_0 = \{F_0, F_1, F_2, \cdots\}$ is countable. Then Player 1 has a move-counting winning strategy if for every reachable vertex v and every $F \in \mathcal{F}_0$*

(i) there exists a v' reachable from v such that F cannot be seen infinitely often from v',
(ii) or there exists some $b \notin F$ and a v_b reachable from v such that b can always be seen again after v_b has been reached.

The idea of the proof is similar to the proof of the previous theorem, as Player 1 can treat the countably many winning sets of Player 0 successively: in her i-th move, she first takes care of F_i by either moving somewhere from where on it cannot be seen infinitely often, or by ensuring that a $b_i \notin F_i$ can always be seen later on. Then, for every $j < i$ which can still be seen infinitely often, she sees the minimal $b_j \notin F_j$ which can be seen infinitely often from all later vertices. This can be done with a move-counting strategy.

▶ **Theorem 4.5.** *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a Muller winning condition where $\mathcal{F}_0 = \{F_0, F_1, F_2, \cdots\}$ is countable. Then $\mathcal{G}$ is determined via move-counting strategies.*

Proof. If the conditions in Lemma 4.4 are satisfied, i.e. if for every reachable v and every $F \in \mathcal{F}_0$, (i) or (ii) holds, then Player 1 has a move-counting winning strategy.

If they are not satisfied, then there exists a reachable v and some $F \in \mathcal{F}_0$ such that after v has been seen, it is always possible to see F infinitely often, and every $b \notin F$ can be prevented from being seen infinitely often. In this case, the argument from Theorem 4.2 suffices to conclude that Player 0 has a move-counting winning strategy. ◀

▶ **Corollary 4.6.** *Let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller winning condition over a countably infinite set of colours where either $\mathcal{F}_0$ or $\mathcal{F}_1$ is countable. Then $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees determinacy via move-counting strategies.*

4.3 Limitations, and length-counting strategies

In the previous sections it was shown that move-counting strategies are useful for interesting classes of Muller winning conditions. One of the reasons why move-counting strategies, where the infinite memory is mostly independent of the actual course of the play, suffice is that these winning conditions are prefix-independent. If non-prefix-independent winning conditions are considered, it is not surprising that move-counting strategies are not always strong enough.

▶ **Example 4.7.** Consider a game on the completely connected graph with vertices 0 and 1, where Player 0 wins those plays where there are infinitely many prefixes that contain more 1s than 0s. Obviously, Player 0 has a winning strategy for this game, simply ensure with every move that afterwards the prefix has the desired property. Still, with a move-counting strategy, this is impossible, and what is more, Player 1 can easily win against every move-counting strategy of Player 0: let g be a move-counting strategy, and let x_n be the number of 1s in the move $g(0, n)$ of the strategy. Player 1 can play in such a way that in her n-th move, after π has already been played, she plays $0^{1+m+x_{n+1}}$, where m is the number of 1s in π. In the resulting play, which is consistent with g, no prefix after the first move of Player 1 has more 1s than 0s, so Player 0 loses and, accordingly, g is not a winning strategy.

One way to partially overcome these sorts of limitations is to consider *length-counting strategies*. Instead of storing the number of the current move, the counter of these strategies stores the length of the current play. Thus, the memory is no longer independent of the play and the occurring moves, but provides information on the distance that has been covered. Still, no data on how this has been done is stored. Notice that the game in the previous example is determined via length-counting strategies. Indeed, the length-counting strategy h where $h(v, n) = 1^{n+1}$ is winning for Player 0. As every move-counting winning strategy can be transformed into a length-counting one, length-counting strategies are strictly stronger than move-counting ones.

▶ **Theorem 4.8.** *Let $\mathcal{G}$ be a Banach-Mazur game that is determined via move-counting strategies. Then it is also determined via length-counting strategies.*

The idea of the proof is as follows. Let g be a move-counting winning strategy of Player σ. Then the length-counting strategy h defined by $h(v, n) := g(v, 0) \cdot g(\_, 1) \cdots g(\_, n)$ is winning for Player σ: any consistent play α is also consistent with g, as the superfluous parts of Player σ's moves could also be part of Player $1 - \sigma$'s moves.

To see that length-counting strategies are not sufficient for Banach-Mazur games in general, we construct a determined Banach-Mazur game which cannot be won using a length-counting strategy.

▶ **Example 4.9.** Consider the graph with vertex set $\omega \cup \omega \times \omega$ and edges $(n, (n, i))$ and $((n, i), n + 1)$ for all $n, i \in \omega$. Thus, every infinite path in this graph (starting at 0) has the form $0 \cdot (0, n_0) \cdot 1 \cdot (1, n_1) \cdot 2 \cdots$ and can be identified with the sequence $n_0 n_1 n_2 \cdots$. Note that a move corresponds to choosing a finite sequence $n_i n_{i+1} \cdots n_{i+k}$. Let the winning paths of Player 0 be these where the sequence $n_0 n_1 n_2 \cdots$ can be split up into disjoint finite sequences $\pi_l = m_0 \cdots m_k$ such that $m_0 = \prod_{j=1}^{k} p_j^{m_j}$, where p_j is the j-th prime (starting with $p_1 = 2$).

It is easy to see that Player 1 can win the described game with her first move by destroying any given sequence of prime factor exponents. However, this cannot be achieved with any length-counting strategy g. As a vertex n can only be reached via paths of length $2n$, such a strategy g assigns a unique move $g(n, 2n)$, i.e. a unique sequence $n_n n_{n+1} \dots n_{n+l}$, to such a vertex, regardless of how it was reached. Consider the play where at positions $j - 1$ (or $(j - 2, n)$) Player 0 chooses a single number N_j, i.e. plays $(j - 1, N_j) \cdot j$, for which $g(j, 2j)$ corresponds to the prime factor exponents. When playing g, Player 1 always answers with $g(j, 2j)$, thus the resulting play is consistent with g, but won by Player 0. As such a play exists for every length-counting strategy of Player 1, the game is not determined via length-counting strategies.

5 Determinacy via FAR strategies

5.1 Muller winning conditions

In the setting of classical graph games, strategies based on finite appearance records have been studied in [8]. In particular, Muller games with countably many colours that are determined via FAR-strategies have been identified. This includes games with Muller conditions $(\mathcal{F}_0, \mathcal{F}_1)$ where $\mathcal{F}_0$ is a singleton, or finite, or a finite union of upwards cones. The proofs in [8] are based on reductions to parity games. In the setting of Banach-Mazur games with Muller winning conditions, these proofs could basically be copied. However, there is also a simple direct proof for the singleton case, reusing the idea for move-counting strategies. For a given winning set F, store the maximal element from F that has been seen already and see, in every move, all elements from F smaller or equal to the stored one, and the next larger one, while simultaneously eliminating the next "bad" colour.

▶ **Theorem 5.1.** *Let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller winning condition over the set ω of colours, such that either $\mathcal{F}_0$ or $\mathcal{F}_1$ is a singleton. Then $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees determinacy via 1-dimensional FAR strategies.*

Notice that for the FAR memory outlined above, the only reachable memory states are the colours in the winning set. If this set is finite, we obtain, by Theorem 1.4, as a direct corollary that positional strategies suffice.

Using a similar idea, combined with the fact that Muller winning conditions over finitely many colours guarantee positional determinacy, one can show that FAR strategies can simulate move-counting strategies (for Muller winning conditions). The maximal colour seen so far is again used as a counter, and in the move for such a memory state n, the moves $g(\_, 0) \cdots g(\_, n)$ of the move-counting strategy are copied, while the counter is also increased.

▶ **Theorem 5.2.** *Let $\mathcal{G} = (G, v_0, \Omega \colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a Muller winning condition with colours in ω that is determined via move-counting strategies. Then it is also determined via 1-dimensional FAR strategies.*

By Theorem 4.5, we obtain as a corollary that many interesting Muller winning conditions guarantee determinacy via FAR-strategies.

▶ **Corollary 5.3.** *Let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller winning condition over the colour set ω such that either $\mathcal{F}_0$ or $\mathcal{F}_1$ is countable. Then $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees determinacy via 1-dimensional FAR strategies.*

5.2 Limitations of FAR strategies

In the above theorem, move-counting strategies for Banach-Mazur games with Muller winning conditions are simulated by FAR strategies. For other winning conditions this is, in general, not possible.

▶ **Example 5.4.** Let G be the completely connected graph with vertices 0 and 1, and let $\mathcal{G}$ be the Banach-Mazur game with arena G where Player 0 wins exactly those plays where every sequence 1^n, $n \in \omega$, is seen infinitely often.

It is easy to see that Player 0 has a move-counting winning strategy, e.g. play 1^n in the n-th move. However, she does not have a positional winning strategy. As every FAR memory on G is finite, and finite memory can be eliminated, this means that she does not have an FAR winning strategy.

Nonetheless, FAR strategies are not weaker than move-counting or even length-counting strategies. To see this, consider the game described in Example 4.9. This game is not determined via length-counting strategies, but via 1-dimensional FAR strategies. It suffices to simply store the maximal priority, i.e. the maximal i of all (n, i) that have been seen. As exponents of prime factors cannot be larger than the actual number, playing two numbers larger than the largest number seen so far is enough to guarantee that Player 1 wins.

In a sense, the classes of games determined via counting strategies and determined via FAR strategies are incomparable. Furthermore, there are games which are not determined via either type of strategy. One example of such a game can be obtained from the game from Example 4.9 by changing the vertex set to $\omega \cup (\omega \times (\omega \setminus \{0\}))$ and by modifying the winning condition so that Player 0 wins exactly those plays where for infinitely many prefixes of the sequence $n_0 n_1 n_2 \cdots$ it holds that $n_k = \prod_{i<k} n_i$. Player 0 has a winning strategy, but neither a length-counting nor an FAR one, as neither type of memory is able to gather enough information about the current prefix to be able to determine the product. Since length-counting strategies only provide information about the length, the actual numbers cannot be reproduced, and as FAR strategies can only store a finite number of previously seen colours, the actual product cannot be computed.

References

1. C. Baier, N. Bertrand, P. Bouyer, T. Brihaye, and M. Größer. Probabilistic and Topological Semantics for Timed Automata. In *Proc. of FSTTCS'07*, volume 4855 of *LNCS*, pages 179–191. Springer, 2007.
2. C. Baier, N. Bertrand, P. Bouyer, T. Brihaye, and M. Größer. Almost-Sure Model Checking of Infinite Paths in One-Clock Timed Automata. In *Proc. of LICS'08*, pages 217–226. IEEE Computer Society, 2008.
3. D. Berwanger, E. Grädel, and S. Kreutzer. Once Upon a Time in the West. Determinacy, Complexity and Definability of Path Games. In *Proc. of LPAR'03*, volume 2850 of *LNCS*, pages 229 – 243. Springer, 2003.
4. A. Blass. Ultrafilters and set theory. In *Ultrafilters across Mathematics*. American Mathematical Society, 2010.
5. J. Büchi and L. Landweber. Solving sequential conditions by finite-state strategies. *Transactions of the AMS*, 138:295–311, 1969.
6. A. Emerson and C. Jutla. Tree automata, mu-calculus and determinacy. In *Proc. 32nd IEEE Symp. on Foundations of Computer Science*, pages 368–377, 1991.
7. E. Grädel. Banach-Mazur Games on Graphs. In *Proc. of FSTTCS'08*, volume 2 of *LIPIcs*, pages 364–382. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2008.

8 E. Grädel and Ł. Kaiser. What kind of memory is needed to win infinitary Muller games? In *Interactive Logic - Selected Papers from the 7th Augustus de Morgan Workshop*, volume 1 of *Texts in Logic and Games*, pages 89 – 116. Amsterdam University Press, 2007.

9 E. Grädel and I. Walukiewicz. Positional Determinacy of Games with Infinitely Many Priorities. *Logical Methods in Computer Science*, 2(4), 2006.

10 A. Kanamori. *The Higher Infinite.* Springer, 1994.

11 A. S. Kechris. *Classical Descriptive Set Theory*, volume 156 of *Graduate Texts in Mathematics.* Springer, 1995.

12 R. D. Mauldin, editor. *The Scottish Book. Mathematics from the Scottish Café.* Birkhäuser, 1981.

13 A. Mostowski. Games with forbidden positions. Tech. Report 78, University of Gdansk, 1991.

14 J. Mycielski, S. Świerczkowski, and A. Zieba. On infinite positional games. *Bull. Acad. Polon. Sci*, 4:485–488, 1956.

15 J. C. Oxtoby. The Banach-Mazur game and Banach category theorem. In *Contributions to the Theory of Games, Vol. III*, volume 39 of *Annals of Mathematics Studies*, pages 159–163. Princeton University Press, 1957.

16 J. C. Oxtoby. *Measure and Category* (2nd ed.). Springer, 1980.

17 M. Pistore and M. Y. Vardi. The Planning Spectrum - One, Two, Three, Infinity. In *Proc. of LICS'03*, pages 234–243. IEEE Computer Society, 2003.

18 R. Telgársky. Topological Games: On the 50th anniversary of the Banach-Mazur game. *Rocky Mountain Journal of Mathematics*, 17(2):227–276, 1987.

19 D. Varacca and H. Völzer. Temporal Logics and Model Checking for Fairly Correct Systems. In *Proc. of LICS'06*, pages 389–398. IEEE Computer Society, 2006.

20 W. Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theoretical Computer Science*, 200(1-2):135–183, 1998.