Locality Theorems in Semiring Semantics

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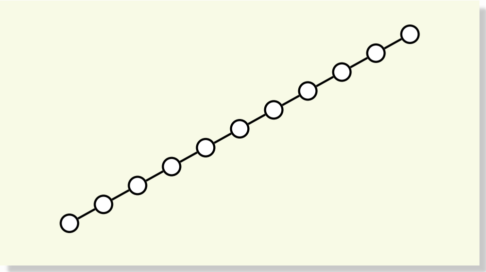
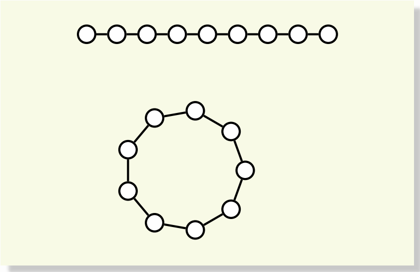
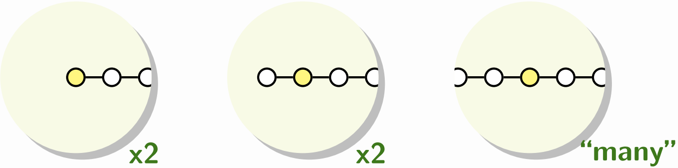
Hanf's Locality Theorem



$\equiv_k$



Isomorphism types of r-neighbourhoods:

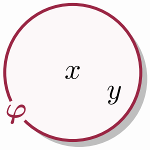


Gaifman Normal Forms

Every FO-formula $\psi(x)$ is equivalent to a Boolean combination of

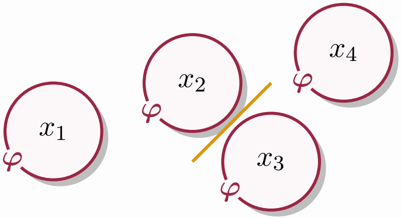
- local formulae, e.g.:

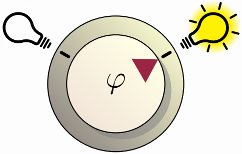
$$\varphi^{(r)}(x) = \exists y(d(x,y) \leq r \ \wedge \ Exy)$$



- basic local sentences:

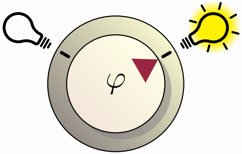
$$\exists x_1 \ldots \exists x_n \Big(\bigwedge_{i<j} d(x_i, x_j) > 2r \ \wedge \ \bigwedge_i \varphi^{(r)}(x_i) \Big)$$





Boolean semantics

Semiring Semantics – An Analogy



Boolean semantics

Semiring semantics

$(K, +, \cdot, 0, 1)$

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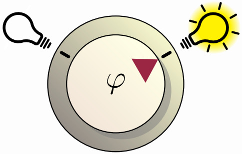

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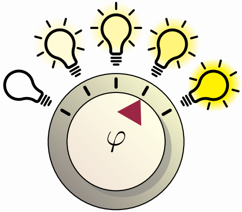


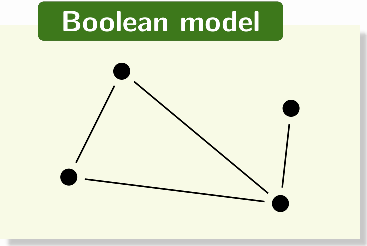
Semiring semantics

$$(K, +, \cdot, 0, 1)$$

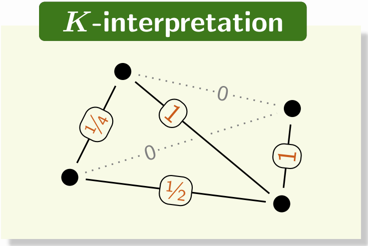
$$(\{0 < \tfrac{1}{4} < \tfrac{1}{2} < \tfrac{3}{4} < 1\}, \max, \min, 0, 1)$$

Semiring Semantics

Idea: Replace Boolean values by semiring values (false: 0, true: $\neq 0$).
Use $+/\max$ to evaluate $\exists, \vee$. Use $\cdot/\min$ for $\forall, \wedge$.



$\rightsquigarrow$



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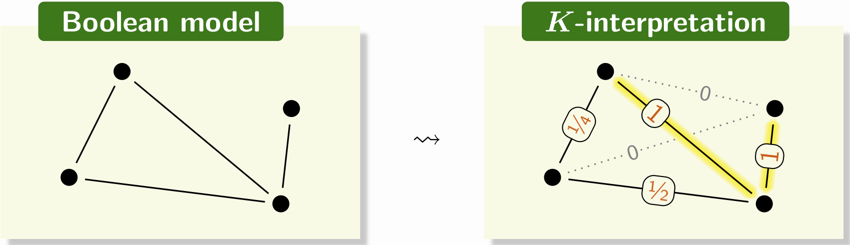
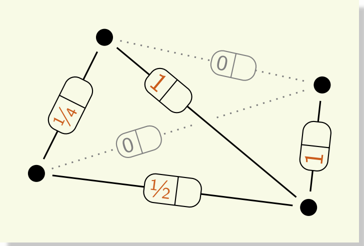


$$[\![\exists x \exists y \exists z (Exy \wedge Eyz)]\!] = \max_{x,y,z \in V} \min([\![Exy]\!], [\![Eyz]\!]) = 1$$

Semiring Semantics – Formal

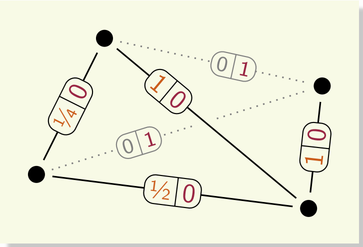


Negation: only in literals

K-interpretation

Assignment π : Literals $\rightarrow K$

Semiring Semantics – Formal



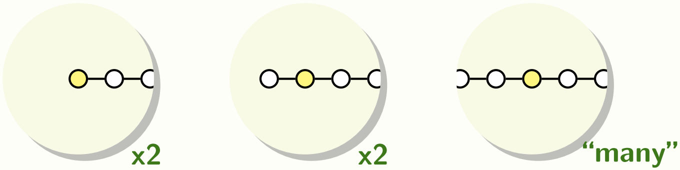
⚠ **Negation:** only in literals

K-interpretation

Assignment π: Literals $\to K$ such that

1. consistency: exactly one of $\pi(Evw)$ and $\pi(\neg Evw)$ is 0,
2. for locality: $\pi(\neg Evw) \in \{0, 1\}$.

Hanf's Theorem in Semirings



Which Semirings?

$$\exists x\, Exx$$

$\mathbb{B}$: truth depends on single witness

K: $\max_x \pi(Exx)$ depends on single witness ✓

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$$\exists x\, Exx$$

$\mathbb{B}$: truth depends on single witness

K: $\max_x \pi(Exx)$ depends on single witness ✓

$\mathbb{N}$: $\sum_x \pi(Exx)$ permits counting of all elements:

$$\sum_{x \in V} \pi(Exx) = \sum_{x \in V} 1 = |V|$$ ✗

⇒ **fully idempotent semirings**

Hanf's Theorem

Theorem

Hanf's Locality Theorem holds in all fully idempotent semirings.
(But not in $\mathbb{N}, \mathbb{T}, \mathbb{R}_+, \ldots$)

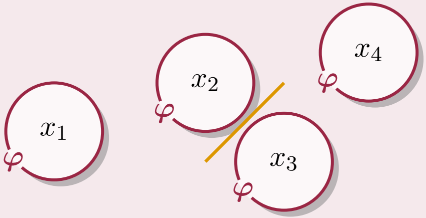
Proof:

- Follow classical proof
- Appropriate notions of $\cong$ and $\equiv_k$ (Grädel, Mrkonjić, ICALP'21)
- Back-and-forth system (EF game) implies $\equiv_k$
  in fully idempotent semirings

Gaifman Normal Forms in Semirings

$$\exists x_1 \dots \exists x_n \Big(\bigwedge_{i<j} d(x_i, x_j) > 2r \;\wedge\; \bigwedge_i \varphi^{(r)}(x_i) \Big)$$

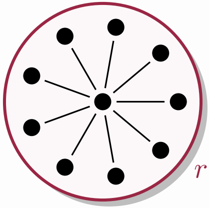
quantifiers relativized
to $d(x_i, y) \leq r$



First Example

$\exists x \forall y\, Exy \quad \equiv_{\mathbb{B}}$

$$\neg \exists x_1 \exists x_2 (d(x_1, x_2) > 2 \ \wedge\ \text{true})$$
$$\wedge\ \exists x_1\ \forall y (d(x_1, y) \leq 2\ \rightarrow\ Ex_1 y)$$



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$$\exists x \forall y\, Exy \quad \equiv_{K} \quad \forall x_1 \forall x_2 (d(x_1, x_2) \leq 2 \;\vee\; \text{false}) \;\wedge\; \exists x_1\, \forall y (d(x_1, y) > 2 \;\vee\; Ex_1 y)$$

The equivalence holds in all semirings.

Normal Forms for Formulae

$$\psi(x) = \exists y(x \neq y \wedge Uy) \quad \equiv_{\mathbb{B}} \quad \exists x_1 \exists x_2 (x_1 \neq x_2 \wedge U x_1 \wedge U x_2) \vee (\neg U x \wedge \exists x_1 \, U x_1)$$

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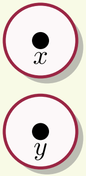
Theorem

No Gaifman normal form of $\psi(x)$ in any naturally-ordered semiring with ≥ 3 elements.



$\pi(Ux) = s$

$\pi(Uy) = t$

$\pi[\![\varphi^{(r)}(x)]\!]$ = polynomial expression in s.

$\pi[\![$ basic local sentence $]\!]$ = symmetric polynomial in s, t.

$\Rightarrow$ **cannot express** $\pi[\![\psi(x)]\!] = t$ **.**

Overview

	$\mathbb{B}$	min-max	fully idempotent	non-idempotent
Hanf	✓	✓	✓	(✗)
Gaifman				
- formulae	✓	✗	✗	✗
- sentences	✓			(✗)

Counterexample

$\exists z \forall x \exists y (Uy \vee x = z)$ has no Gaifman normal form in the Tropical semiring.

Normal Forms for Sentences

Main Result

Gaifman normal forms exist for sentences in min-max semirings.

Every sentence is equivalent to a positive Boolean combination of basic local sentences

$$\exists \mathbf{x}\Big(\bigwedge_{i<j} d(x_i, x_j) > 2r \wedge \bigwedge_i \varphi^{(r)}(x_i)\Big) \quad \text{and} \quad \forall \mathbf{x}\Big(\bigvee_{i<j} d(x_i, x_j) \leq 2r \vee \bigvee_i \varphi^{(r)}(x_i)\Big).$$

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Remarks:

- Textbook proofs not applicable (characteristic sentences)
- Gaifman's proof (1982): quantifier elimination, case distinctions $\varphi \wedge (C \vee \neg C)$
- Here: elimination of quantifier alternations, surprisingly difficult

Full Circle

work in progress

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Standard Semantics

Every sentence has a Gaifman normal form which does not add negations.

+

Semirings

Lift to min-max/lattice semirings via separating homomorphisms.

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Recall:

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Summary

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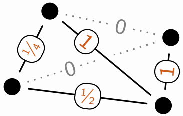

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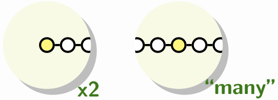

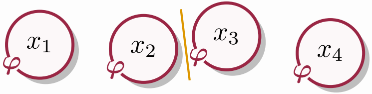

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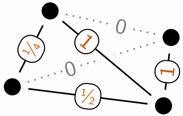

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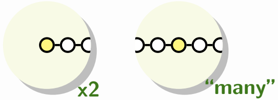

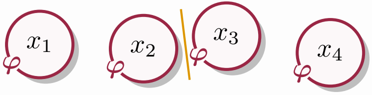

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Thank you for listening