On the Presburger Fragment of Logics with Multiteam Semantics

Richard Wilke

Modern logics for arguing about dependence and independence are based on team semantics [Vää07]. From a purely logical point of view these logics have clean theoretical properties, as for example inclusion and exclusion logic corresponds to independence logic which again is equivalent to existential second-order logic. However, in these logics data is represented as teams which are *sets of assignments*, hence one can only argue about the *presence or absence* of data. As in many real world applications the *multiplicities* are a key factor (e.g. in databases) different logics that incorporate such information have been proposed [HPV15, HPV17, DHK+18]. In this article we consider *multisets of assignments*, called *multiteams*, which extend teams by the number of occurrences of each assignment. Notions such as independence in this setting only make sense if the multiplicities are natural numbers, hence we consider only finite multiteams and structures.

Logics with team semantics without negation are embeddable in existential second-order logic Σ^1_1, similarly logics with multiteam semantics can be embedded into the second-order logic $\mathrm{ESO}^{\mathsf{mts}}[+,\cdot]$ with built-in features for dealing with arithmetic. Formally, all structures are extended by a numerical sort and second-order quantifiers over functions $f : A^k \longrightarrow \mathbb{N}$ mapping tuples of elements of the universe A of a structure $\mathfrak{A}$ to natural numbers are added to first-order logic. Additionally, basic arithmetic $+$ and $\cdot$ is available allowing terms of the kind $f\bar{x} + g\bar{y}$. This note intends to discuss the Presburger fragment $\mathrm{ESO}^{\mathsf{mts}}[+]$ of this logic, i.e. the restriction where only addition is allowed, but no multiplication. As it has turned out on the level of multiteam semantics this logic is equivalent to $\mathrm{FO}^{\mathsf{M}}[\Subset, \mid\,]$, that is multiteam inclusion / exclusion logic. The focus of the present work is on multiteam logics, and due to the space limitations we will not investigate the second-order logic $\mathrm{ESO}^{\mathsf{mts}}[+]$; for the same reason most proofs are omitted, sometimes when we translate a logic into another we present the formula that expresses an atom of one logics in the other, but do not argue for its correctness. We aim at discussing the logic $\mathrm{FO}^{\mathsf{M}}[\Subset, \mid\,]$ in more detail which includes finding an atom α such that $\mathrm{FO}^{\mathsf{M}}[\Subset, \mid\,] \equiv \mathrm{FO}^{\mathsf{M}}[\alpha]$. In team semantics independence logic is equivalent to inclusion / exclusion logic, which – as we will discover – is not the case under multiteam semantics, but multiteam inclusion can still express both multiteam inclusion and exclusion. It turns out that in multiteam logics with a *forking* atom $\lessdot_{=1/2}$ both inclusion and exclusion are expressible. This leads to a further analysis of the different variants of forking $\lessdot_{\leq p}$ and $\lessdot_{\geq p}$ for some $p \in [0, 1]$.

§1 Multiteam Semantics

A multiset $M = (S, n)$ is a tuple of a (finite) set S together with a function $n : S \longrightarrow \mathbb{N}_{>0}$ assigning every element its *multiplicity*. The additive union of two multisets $(S, n) \uplus (S', n')$ is $(S \cup S', n + n')$. Inclusion $M \Subset M'$ means that $n(s) \leq n'(s)$ holds for all $s \in S$. For a number k the multiple kM is $\biguplus_{i<k} M$. A *multiteam* M is a multiset (X, n) such that X is a team. We fix some notation. The *support*, or *underlying team*, of M is $M^{\mathsf{T}} := X$; the evaluation of M on a tuple $\bar{x}$, written $M(\bar{x})$, is the multiset $\{\!\{ s(\bar{x}) : s \in M \}\!\}$, where $\{\!\{ \lambda(s) : s \in M \}\!\}$ is a notation for $\biguplus_{s \in M^{\mathsf{T}}} n(s) \{\!\{ \lambda(s) \}\!\}$; the *restriction* $M\restriction_\rho$ is $\{\!\{ s \in M : s \vDash \rho \}\!\}$; the *probability* $\Pr_M(\bar{x} = \bar{a})$ that the variable $\bar{x}$ takes value $\bar{a}$ in M is defined as $|M\restriction_{\bar{x}=\bar{a}}|/|M|$, and the conditional probability $\Pr_M(\bar{x} = \bar{a} \mid \bar{y} = \bar{b})$ is defined similarly. Moreover, $\zeta \longrightarrow \xi$ is a shorthand for $\neg\zeta \vee (\zeta \wedge \xi)$.

The dependency concepts known from team semantics or database theory can be understood in a natural way under multiteam semantics. Further, the access to multiplicities gives rise to additional notions. The following lists the most important ones that are considered throughout this abstract.

Definition 1 (Multidependence Atoms)**.** Let $\mathfrak{A}$ be a finite structure and M a multiteam.

Dependence: $\mathfrak{A} \models_M \mathrm{dep}(\bar{x}, y) :\iff \mathfrak{A} \models_{M^\mathsf{T}} \mathrm{dep}(\bar{x}, y)$
Exclusion: $\mathfrak{A} \models_M \bar{x} \mid \bar{y} :\iff \mathfrak{A} \models_{M^\mathsf{T}} \bar{x} \mid \bar{y}$
Inclusion: $\mathfrak{A} \models_M \bar{x} \Subset \bar{y} :\iff M(\bar{x}) \Subset M(\bar{y})$
Statistical independence: $\mathfrak{A} \models_M \bar{x} \perp\!\!\!\perp \bar{y}$ holds if, and only if, $\Pr_M(\bar{x} = \bar{a}) = \Pr_M(\bar{x} = \bar{a} \mid \bar{y} = \bar{b})$ for all $\bar{a} \in M(\bar{x})$ and $\bar{b} \in M(\bar{y})$. An equivalent condition is that $M(\bar{x}) \times M(\bar{y}) = |M| \cdot M(\bar{x}\bar{y})$.
Conditional independence: $\mathfrak{A} \models_M \bar{x} \perp\!\!\!\perp_{\bar{z}} \bar{y}$ if $\Pr_M(\bar{x} = \bar{a} \mid \bar{z} = \bar{c}) = \Pr_M(\bar{x} = \bar{a} \mid \bar{y}\bar{z} = \bar{b}\bar{c})$ for all $\bar{a} \in A^{|\bar{x}|}$, $\bar{b} \in A^{|\bar{y}|}$ and $\bar{c} \in A^{|\bar{z}|}$. ⊿

First-order operators can be defined as either being *strict*, i.e. using each assignment exactly once, or *lax*. In team semantics lax operators turned out to be the correct choice, which intuitively is based on the fact that only the information whether or not an assignment is present is available in a team. The situation is different under multiteam semantics since the multiplicities are accessible and an analysis has shown that indeed strict semantics should be assumed [GW]. For a set of multiteam dependency notions Ω, its closure under first-order operators is denoted by $\mathrm{FO}^\mathsf{M}[\Omega]$.

Definition 2 (Multiteam Semantics)**.** Multiteam semantics is defined by the following rules. Let Ω be a set of multidependency atoms, $\mathfrak{A}$ a structure, M a multiteam over A and $\psi, \psi_1, \psi_2 \in \mathrm{FO}^\mathsf{M}[\Omega]$.

- $\mathfrak{A} \models_M \psi_1 \wedge \psi_2$ if $\mathfrak{A} \models_M \psi_1$ and $\mathfrak{A} \models_M \psi_2$;
- $\mathfrak{A} \models_M \psi_1 \vee \psi_2$ if there are $M_1 \uplus M_2 = M$ with $\mathfrak{A} \models_{M_i} \psi_i$;
- $\mathfrak{A} \models_M \forall x \psi$ if $\mathfrak{A} \models_{M[x \mapsto A]} \psi$;
- $\mathfrak{A} \models_M \exists x \psi$ if $\mathfrak{A} \models_{M[x \mapsto F]} \psi$ for some function $F : M \longrightarrow A$.

Where $M[x \mapsto A] = \{\!\{ s[x \mapsto a] : s \in M, a \in A \}\!\}$, i.e. every assignment in M is updated with every value of A, thus $|M[x \mapsto A]| = |A| \cdot |M|$. The function F maps every assignment $s \in M$ to a value $F(s) \in A$. If an assignment s is present more than once in M each copy may or may not receive a different value from F. Accordingly $M[x \mapsto F]$ denotes $\{\!\{ s[x \mapsto F(s)] : s \in M \}\!\}$, especially $|M[x \mapsto F]| = |M|$. ⊿

Downwards- and union closure are defined analogously to team semantics, i.e. φ is downwards closed if $\mathfrak{A} \models_M \varphi$ implies $\mathfrak{A} \models_R \varphi$ for all $R \Subset M$ and ψ is union closed in case $\mathfrak{A} \models_M \psi$ and $\mathfrak{A} \models_R \psi$ implies $\mathfrak{A} \models_{M \uplus R} \psi$. To avoid confusion between team and multiteam semantics we write FO^T for first-order team logic and accordingly FO^M for first-order multiteam logic.

§2 Between Inclusion, Exclusion and Independence

Let us start by repeating the picture in team semantics. Independence logic $\mathrm{FO}^\mathsf{T}[\perp]$ and conditional independence logic $\mathrm{FO}^\mathsf{T}[\perp_\mathsf{c}]$ coincide, as was shown by Galliani [Gal12]. The proof provides translations of exclusion and inclusion atoms into independence logic and a formula that expresses conditional independence by means of inclusion / exclusion, i.e. $\mathrm{FO}^\mathsf{T}[\perp_\mathsf{c}] \leq \mathrm{FO}^\mathsf{T}[\subseteq, \mid\,] \leq \mathrm{FO}^\mathsf{T}[\perp]$ and hence $\mathrm{FO}^\mathsf{T}[\perp] \equiv \mathrm{FO}^\mathsf{T}[\perp_\mathsf{c}] \equiv \mathrm{FO}^\mathsf{T}[\subseteq, \mid\,]$. We observe that instead of going through this chain of translations, conditional independence can be defined by using just a single independence atom in team semantics.

Example 3. The formula $\varphi_{\perp_\mathsf{c}}(\bar{x}, \bar{y}, \bar{z}) \in \mathrm{FO}^\mathsf{T}[\perp]$ is equivalent to $\bar{x} \perp_{\bar{z}} \bar{y}$, where

$$\varphi_{\perp_\mathsf{c}}(\bar{x}, \bar{y}, \bar{z}) := \forall \bar{p} \exists \bar{u} \exists \bar{w} \big((\bar{z} = \bar{p} \longrightarrow \bar{u}\bar{w} = \bar{x}\bar{y}) \wedge (\bar{z} \neq \bar{p} \vee \bar{z}\bar{u} \perp \bar{p}\bar{w}) \big).$$

Intuitively this formula builds from a given multiteam M an extension M' such that $M'\!\upharpoonright_{\bar{p}=\bar{z}=\bar{a}}(\bar{u}, \bar{w}) = M\!\upharpoonright_{\bar{z}=\bar{a}}(\bar{x}, \bar{y})$. Further, no restriction on M' is imposed whenever $\bar{p}$ and $\bar{z}$ differ, hence all possible combinations may be present which implies that $M' \models \bar{z}\bar{u} \perp \bar{p}\bar{w}$ holds if and only if $M \models \bar{x} \perp_{\bar{z}} \bar{y}$. ◊

A similar technique however fails under multiteam semantics. Nevertheless for the special case of dependence $\mathrm{dep}(\bar{x}, y)$, which is equivalent to $y \perp_{\bar{x}} y$ and $y \perp\!\!\!\perp_{\bar{x}} y$, this idea is applicable for multiteams.

Proposition 4. $\mathrm{FO}^{\mathsf{M}}[\mathrm{dep}] \leq \mathrm{FO}^{\mathsf{M}}[\perp\!\!\!\perp]$.

Proof. As already stated $\mathrm{dep}(\bar{x}, y) \equiv y \perp\!\!\!\perp_{\bar{x}} y$ which we claim to be equivalent to

$$\psi := \forall \bar{p} \exists u \big((\bar{x} = \bar{p} \longrightarrow u = y) \wedge \bar{x} y \perp\!\!\!\perp \bar{p} u \big).$$

Assume $\mathfrak{A} \not\models \mathrm{dep}(\bar{x}, y)$. Thus there are $s, s' \in M^{\mathsf{T}}$ with $s(\bar{x}) = s'(\bar{x}) = \bar{a}$ but $b = s(y) \neq s'(y) = c$. Towards a contradiction assert that $\mathfrak{A} \models_M \psi$. Let R be the multiteam $M[\bar{p} \mapsto A^k][u \mapsto F]$ for an appropriate F such that $R \models (\bar{x} = \bar{p} \longrightarrow u = y)$. Observe that $0 = \Pr_R(\bar{x}y = \bar{a}b \mid \bar{p}u = \bar{a}c) < \Pr_R(\bar{x}y = \bar{a}b)$.

On the other hand assume $\mathfrak{A} \models_M \mathrm{dep}(\bar{x}, y)$, i.e. $f: M^{\mathsf{T}}(\bar{x}) \longrightarrow A$ exists such that for all assignments $s \in M$ holds $s(y) = f(s(\bar{x}))$. We describe how the values for u can be chosen such that we witness $\mathfrak{A} \models_M \psi$. The choice is clear for all assignments in which $\bar{x}$ and $\bar{p}$ agree. If $s(\bar{x}) \neq s(\bar{p})$ we put $s(u) = f(s(\bar{p}))$, in case $s(\bar{p})$ is a value occurring in $M(\bar{x})$ and $s(u) = c$ for an arbitrary fixed value $c \in A$ otherwise. Let S be the resulting multiteam. Certainly $\Pr_S(\bar{x}y = \bar{a}b) = \Pr_S(\bar{x}y = \bar{a}b \mid \bar{p}u = \bar{c}d)$ for all $\bar{a}, \bar{c}, b, d \in A^*$. □

In our favour we can prove that multiteam inclusion is expressible in statistical independence logic by a slight modification of the formula that defines inclusion via independence in team semantics [Gal12].

Proposition 5. $\mathrm{FO}^{\mathsf{M}}[\Subset] \leq \mathrm{FO}^{\mathsf{M}}[\perp\!\!\!\perp]$.

Proof. Without going into detail we claim that $\bar{x} \Subset \bar{y}$ is equivalent to the formula $\varphi_{\Subset}(\bar{x}, \bar{y})$, where $\varphi_{\Subset}(\bar{x}, \bar{y}) := \forall a, b, \bar{z} \big((\bar{z} \neq \bar{x} \wedge \bar{z} \neq \bar{y}) \vee (\bar{z} \neq \bar{y} \wedge a \neq b) \vee (\bar{z} = \bar{y} \wedge a = b) \vee ((\bar{z} = \bar{y} \vee a = b)) \wedge \bar{z} \perp\!\!\!\perp ab \big)$. □

Hence, similarly to team semantics multiteam independence can express dependence (and hence exclusion) and inclusion. However, multiteam inclusion and exclusion logic is less expressive than statistical independence logic as is demonstrated by the upcoming theorem that we state without proof (cf. [GW]). In fact we find that no combination of downwards closed and union closed atomic dependency notions is able to express independence under multiteam semantics.

Theorem 6. Let $\bar{\alpha}$ be any collection of downwards closed atoms and $\bar{\beta}$ be any collection of union closed atoms. There is no formula $\psi \in \mathrm{FO}^{\mathsf{M}}[\bar{\alpha}, \bar{\beta}]$ with $x \perp\!\!\!\perp y \equiv \psi(x, y)$.

This leaves open two questions, first: Can statistical independence logic $\mathrm{FO}^{\mathsf{M}}[\perp\!\!\!\perp]$ express conditional independence $\bar{x} \perp\!\!\!\perp_{\bar{z}} \bar{y}$? As the previous statements demonstrate the methods applicable in team semantics that show $\mathrm{FO}^{\mathsf{T}}[\perp] \equiv \mathrm{FO}^{\mathsf{T}}[\perp_{\mathsf{c}}]$ do not translate to multiteam semantics. The second issue is whether there is a (natural) atomic formula α such that $\mathrm{FO}^{\mathsf{M}}[\Subset, |\,] \equiv \mathrm{FO}^{\mathsf{M}}[\alpha]$? We leave open the first question and give a positive answer to the second question in the following paragraph.

§3 Forking / Anonymity

We now turn our attention to atoms that can count assignments by their *forking degree*, i.e. such an atom may state that depending on a variable the values of another one all occur with probability at least (at most) a given threshold. Grädel and Hegselmann [GH16] investigated the notion of forking in context of team semantics. To do so they augmented each structure by a number sort which enabled writing formulae such as $x \lhd^{\leq \lambda} y$ which states that depending on x at most λ different values for y occur, where λ is a variable over the number sort. Since handling natural numbers is a built-in feature of multiteam semantics we do not need to consider two sorted structures in order to define a meaningful concept. For technical reasons we assume all structures to contain at least two elements in the following.

Definition 7 (Forking)**.** Let M a multiteam over some finite set A and $p \in [0,1]$. For $\triangleleft \in \{\leq, =, \geq\}$ the *forking atom* $\lessdot_{\triangleleft p}$ is defined via $M \models \bar{x} \lessdot_{\triangleleft p} \bar{y}$, if $\Pr_M(\bar{y} = \bar{b} \mid \bar{x} = \bar{a}) \triangleleft p$ for all $\bar{a}, \bar{b} \in A^*$. ⊿

The forking atom $\lessdot_{\leq 1/2}$ resembles a multiteam version of the anonymity atom that was introduced by Väänänen [GKKV19]. It states that the values a certain variable takes do not suffice to determine the value of another. More formally, $x \Upsilon y$ is satisfied in a team X whenever for every value a that x takes in X there are (at least) two assignments s and s' such that $s(x) = s'(x) = a$ but $s(y) \neq s'(y)$. This atom is in fact equivalent to non-dependence [Gal15]. In multiteam semantics we may further impose the degree p of anonymity in $\lessdot_{\leq p}$ giving us a natural atom defining the concept of anonymity.

Let us start the analysis of the forking atoms by examining the closure properties of the different forking variants.

Proposition 8. Let $p \in (0,1)$, $q \in (0, 1/2]$ and $r \in \{\frac{1}{n} : n > 1\}$. $\lessdot_{\leq p}$ is union- but not downwards closed, while $\lessdot_{\geq q}$ and $\lessdot_{=r}$ are neither of both.

There are two conspicuousnesses of this proposition. First, the thresholds for which the statements hold exclude certain cases. For some of these values the forking atoms trivialise; indeed we observe that $\bar{x} \lessdot_{\leq 0} y \equiv \bar{x} \lessdot_{=0} y \equiv \mathsf{false}$, $\bar{x} \lessdot_{\leq 1} y \equiv \bar{x} \lessdot_{\geq 0} y \equiv \mathsf{true}$ and furthermore $\bar{x} \lessdot_{=p} \bar{y} \equiv \mathsf{false}$ for all $p \neq 1/n$. The remaining atoms, i.e. $\bar{x} \lessdot_{=1} \bar{y}$ and $\bar{x} \lessdot_{\geq p} \bar{y}$ for $p > 1/2$, all coincide with the dependence atom $\mathrm{dep}(\bar{x}; \bar{y})$.

This explains the choice of the thresholds. Secondly, one might expect a symmetry between $\lessdot_{\leq p}$ and $\lessdot_{\geq p}$ like, for example, one being union closed and the other downwards closed. While this is not true we find that $\lessdot_{\geq p}$ is in fact *weakly downwards closed* for all $p \in [0,1]$.

Definition 9. A formula φ is *weakly downwards closed*, or *downwards closed in the team semantical sense*, if $\mathfrak{A} \models_{(X,n)} \varphi$ implies $\mathfrak{A} \models_{(Y,m)} \varphi$ for all $(Y, m) \Subset (X, n)$ such that $n(s) = m(s)$ for all $s \in Y$. ⊿

Since the other forking atoms are not weakly downwards closed we obtain the following relationship.

Corollary 10. The logics $\mathrm{FO}^{\mathsf{M}}[\lessdot_{\leq p}]$ and $\mathrm{FO}^{\mathsf{M}}[\lessdot_{\geq q}]$ are incomparable for all $p \in (0,1)$ and $q \in (0,1]$.

Let us continue our analysis by comparing forking logics to more well known logics with multiteam semantics. Because of the severe space limitations and since the formulae arising in the upcoming proofs are too long and difficult to parse we state the relationships without presenting even the formulae used in the translations. However, we hope that the closure properties provide enough intuition for the reader to believe the statements.

Theorem 11.
(1) $\mathrm{FO}^{\mathsf{M}}[\mathrm{dep}] \leq \mathrm{FO}^{\mathsf{M}}[\lessdot_{\geq 1/2}], \mathrm{FO}^{\mathsf{M}}[\lessdot_{=1/2}]$.
(2) $\mathrm{FO}^{\mathsf{M}}[\Subset] \leq \mathrm{FO}^{\mathsf{M}}[\lessdot_{\leq 1/2}], \mathrm{FO}^{\mathsf{M}}[\lessdot_{=1/2}]$.
(3) $\mathrm{FO}^{\mathsf{M}}[\lessdot_{\leq 1/2}] \leq \mathrm{FO}^{\mathsf{M}}[\Subset]$.

Corollary 12. $\mathrm{FO}^{\mathsf{M}}[\lessdot_{\leq 1/2}] \equiv \mathrm{FO}^{\mathsf{M}}[\Subset]$.

This enables us to identify $\lessdot_{=1/2}$ as the atom equivalent to inclusion / exclusion in multiteam semantics.

Theorem 13. $\mathrm{FO}^{\mathsf{M}}[\lessdot_{\geq 1/2}] \lneq \mathrm{FO}^{\mathsf{M}}[\mathrm{dep}, \Subset] \equiv \mathrm{FO}^{\mathsf{M}}[\lessdot_{=1/2}]$.

Proof. By Theorem 11 we may use $\lessdot_{\leq 1/2}$ as it is available in $\mathrm{FO}^{\mathsf{M}}[\Subset]$. Then $\bar{x} \lessdot_{=1/2} \bar{y} \equiv (\mathrm{dep}(\bar{x}, \bar{y}) \vee \mathrm{dep}(\bar{x}, \bar{y})) \wedge \bar{x} \lessdot_{\leq 1/2} \bar{y}$.

$\bar{x} \lessdot_{\geq 1/2} \bar{y} \equiv \mathrm{dep}(\bar{x}, \bar{y}) \vee_{\bar{x}} \bar{x} \lessdot_{=1/2} \bar{y}$, where $\mathfrak{A} \models_M \varphi \vee_{\bar{x}} \psi \iff$ there are $R \uplus S = M$ such that $\mathfrak{A} \models_R \varphi$, $\mathfrak{A} \models_S \psi$ and for all $s, s' \in M$ if $s(\bar{x}) = s'(\bar{x})$ then both s and s' belong either to R or to S. It is easy to define this kind of disjunction using dependence atoms. □

Notice that $\mathrm{FO}^{\mathsf{M}}[\lessdot_{=1/2}] \equiv \mathrm{FO}^{\mathsf{M}}[\lessdot_{\leq 1/2}, \lessdot_{\geq 1/2}]$ follows as a corollary. Let us end this section by demonstrating that using $\lessdot_{=1/2}$ one can express $\lessdot_{=1/n}$ for all $n \in \mathbb{N}_{>0}$.

Proposition 14. $\mathrm{FO}^{\mathsf{M}}[\lhd_{=1/n}] \leq \mathrm{FO}^{\mathsf{M}}[\lhd_{=1/2}]$ for all $n \in \mathbb{N}_{>0}$.

Proof. The special case $n = 1$ was handled in Theorem 11 which also allows us to make use of dependence atoms. Let $n > 1$ (of course $n = 2$ is trivial but also covered by the upcoming construction). We claim that $\bar{x}\lhd_{=1/n}\bar{y}$ is equivalent to the formula η:

$$\exists \bar{y}_1 \cdots \exists \bar{y}_n(\bigwedge_{i<n} \mathrm{dep}(\bar{x}, \bar{y}_i) \wedge \bigwedge_{i \neq j} \bar{y}_i \neq \bar{y}_j \wedge (\bigvee_{i<n} \bar{y} = \bar{y}_i) \wedge \bigwedge_{i \neq j}(\bar{y} = \bar{y}_i \vee \bar{y} = \bar{y}_j \longrightarrow \bar{x}\lhd_{=1/2}\bar{y})).$$

Before we start the analysis, notice that the formula η is $\bar{x}$-guarded, that is $\mathfrak{A} \models_M \eta$ holds if, and only if, $\mathfrak{A} \models_{M\restriction_{\bar{x}=\bar{a}}} \eta$ for all $\bar{a} \in M^T(\bar{x})$. In fact, instead of η one may consider its unguarded version, that is the formula η, where $\mathrm{dep}(\bar{x}, \bar{y}_i)$ is exchanged by $\mathrm{dep}(\bar{y}_i)$ and $\bar{x}\lhd_{=1/2}\bar{y}$ by $\lhd_{=1/2}\bar{y}$. Since $\bar{x}\lhd_{=1/n}\bar{y}$ is also $\bar{x}$-guarded we will for the sake of simplicity in the following consider $\lhd_{=1/n}\bar{y}$ and multiteams M with domain $\bar{y}$. Assume $\mathfrak{A} \models_M \lhd_{=1/n}\bar{y}$. Thus, $\Pr_M(\bar{y} = \bar{b}) = 1/n$ for each $\bar{b} \in M^T(\bar{y})$, implying that $|M^T(\bar{y})| = n$. Let us write this set as $\{\bar{b}_1, \dots, \bar{b}_n\}$. To show the claim $\mathfrak{A} \models_M \eta$, let M' be the extension of M by values for $\bar{y}_1, \dots, \bar{y}_n$ such that for all $s \in M^T$ holds $s(\bar{y}_i) = \bar{b}_i$. By construction $\mathfrak{A} \models_{M'} \bigwedge_{i<n} \mathrm{dep}(\bar{y}_i) \wedge \bigwedge_{i \neq j} \bar{y}_i \neq \bar{y}_j \wedge (\bigvee_{i<n} \bar{y} = \bar{y}_i)$. Hence it remains to verify $\mathfrak{A} \models_{M'} \bigwedge_{i \neq j}(\bar{y} = \bar{y}_i \vee \bar{y} = \bar{y}_j \longrightarrow \lhd_{=1/2}\bar{y})$, which is the case if for all $i \neq j$ holds $\Pr_R(\bar{y} = \bar{b}_i) = \Pr_R(\bar{y} = \bar{b}_j) = 1/2$ where $R = M'\restriction_{\bar{y} \in \{\bar{b}_i, \bar{b}_j\}}$. This is equivalent to $|M'\restriction_{\bar{y}=\bar{b}_i}| = |M'\restriction_{\bar{y}=\bar{b}_j}|$. By assumption the probability that $\bar{y}$ takes any value equals $1/n$. Thus all values for $\bar{y}$ must be equally distributed whence $|M'\restriction_{\bar{y}=\bar{b}_i}| = |M'\restriction_{\bar{y}=\bar{b}_j}|$ and hence $\mathfrak{A} \models_M \eta$ follows.

Conversely let $\mathfrak{A} \models_M \eta$. Thus there is an extension M' of M by (constant) values $\bar{b}_1, \dots, \bar{b}_n$ for $\bar{y}_1$ through $\bar{y}_n$ such that $M' \models \bigwedge_{i<n} \mathrm{dep}(\bar{y}_i) \wedge \bigwedge_{i \neq j} \bar{y}_i \neq \bar{y}_j \wedge (\bigvee_{i<n} \bar{y} = \bar{y}_i)$. Hence $|M^T(\bar{y})| \leq n$. Moreover $M' \models \bigwedge_{i \neq j}(\bar{y} = \bar{y}_i \vee \bar{y} = \bar{y}_j \longrightarrow \lhd_{=1/2}\bar{y})$, implying that for all $i \neq j$ we have $|M\restriction_{\bar{y}=\bar{b}_i}| = |M\restriction_{\bar{y}=\bar{b}_j}|$. Therefore $|M^T(\bar{y})| = n$ (if there are less than n values one of these multisets is empty and hence not equivalent to another non empty one, which must exist). Since M contains the same amount of assignments that map $\bar{y}$ to $\bar{b}_i$ as those that map $\bar{y}$ to $\bar{b}_j$ we conclude that $\Pr_M(\bar{y} = \bar{b}_i) = 1/n$ for $i \in \{1, \dots, n\}$. ☐

§4 Summary

Figure 1 displays the relationships of the various logics considered in this note and shows the corresponding relations for logics with team semantics.

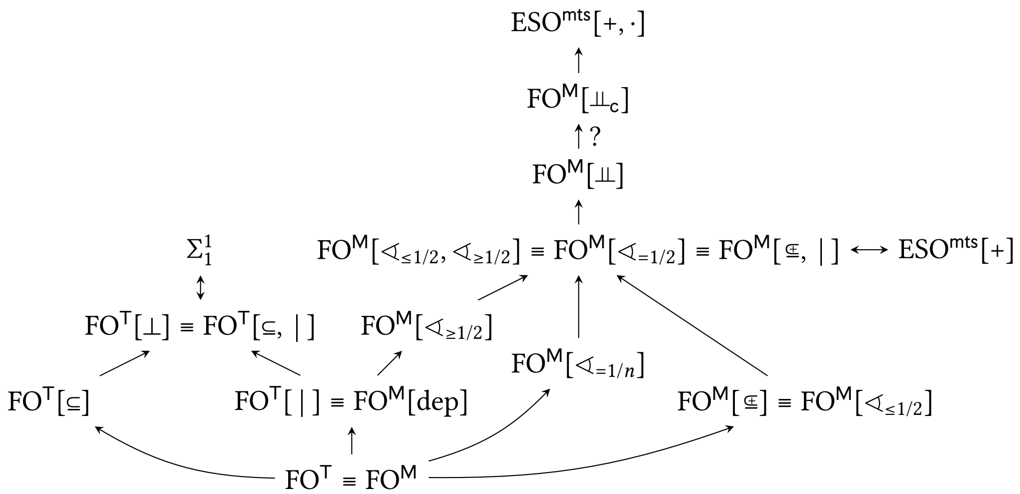

Figure 1: An arrow $L \longrightarrow R$ means $L \lneq R$ and $L \leftrightarrow R$ stands for $L \equiv R$. The precise relationship between statistical independence and conditional independence logic remains open.

References

[DHK+18] A. Durand, M. Hannula, J. Kontinen, A. Meier, and J. Virtema. Approximation and dependence via multiteam semantics. *Annals of Mathematics and Artificial Intelligence*, 2018.

[Gal12] P. Galliani. Inclusion and exclusion in team semantics — on some logics of imperfect information. *Annals of Pure and Applied Logic*, 163:68–84, 2012.

[Gal15] Pietro Galliani. Upwards closed dependencies in team semantics. *Inf. Comput.*, 245:124–135, 2015.

[GH16] Erich Grädel and Stefan Hegselmann. Counting in Team Semantics. In Jean-Marc Talbot and Laurent Regnier, editors, *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, volume 62 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 35:1–35:18, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[GKKV19] Erich Grädel, Phokion G. Kolaitis, Juha Kontinen, and Heribert Vollmer. Logics for Dependence and Independence (Dagstuhl Seminar 19031). *Dagstuhl Reports*, 9(1):28–46, 2019.

[GW] Erich Grädel and Richard Wilke. Logics with multiteam semantics. To appear.

[HPV15] T. Hyttinen, G. Paolini, and J. Väänänen. Quantum team logic and bell's inequalities. *Review of Symbolic Logic*, 8:722–742, 2015.

[HPV17] T. Hyttinen, G. Paolini, and J. Väänänen. A logic for arguing in measure teams. *Archive of Mathematical Logic*, 56:475–489, 2017.

[Vää07] J. Väänänen. *Dependence logic: A new approach to independence friendly logic*, volume 70. Cambridge University Press, 2007.