Invariants of Automatic Presentations and Semi-Synchronous Transductions with Appendix

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Abstract. Automatic structures are countable structures finitely presentable by a collection of automata. We study questions related to properties invariant with respect to the choice of an automatic presentation. We give a negative answer to a question of Rubin concerning definability of intrinsically regular relations by showing that order-invariant first-order logic can be stronger than first-order logic with counting on automatic structures. We introduce a notion of equivalence of automatic presentations, define semi-synchronous transductions, and show how these concepts correspond. Our main result is that a one-to-one function on words preserves regularity as well as non-regularity of all relations iff it is a semi-synchronous transduction. We also characterize automatic presentations of the complete structures of Blumensath and Grädel.

1 Introduction

Automatic structures are countable structures presentable by a tuple of finite automata. Informally, a structure is automatic if it is isomorphic to one consisting of a regular set of words as universe and having only regular relations, i.e., which are recognizable by a finite synchronous multi-tape automaton. Every such isomorphic copy, and any collection of automata representing it, as well as the isomorphism itself may, and will, ambiguously, be called an *automatic presentation* (a.p.) of the structure. It follows from basic results of automata theory [9] that the first-order theory of every automatic structure is decidable. Using automata on ω-words or finite- or infinite trees in the presentations, one obtains yet other classes of structures with a decidable first-order theory [4, 6]. This paper is solely concerned with presentations via automata on finite words.

The notion of (ω-)automatic structures first appeared in [12]. Khoussainov and Nerode [13] have reintroduced and elaborated the concept, and [5] has given momentum to its systematic investigation. Prior to that, automatic groups [8], automatic sequences [1], and expansions of Presburger arithmetic [3, 7] by relations regular in various numeration systems have been studied thoroughly. These

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investigations concern only certain naturally restricted automatic presentations of the structures involved.

The first logical characterization of regularity is due to Büchi. Along the same lines Blumensath and Grädel characterized automatic structures in terms of interpretations [4, 6]. They have shown that for each finite non-unary alphabet Σ, $\mathfrak{S}_\Sigma = (\Sigma^*, (S_a)_{a\in\Sigma}, \preceq, \mathsf{el})$ is a complete automatic structure, i.e. such that all automatic structures, and only those, are first-order interpretable in it. (See Example 1 and Prop. 2 below.) In this setting each interpretation, as a tuple of formulas, corresponds to an a.p. given by the tuple of corresponding automata.

An aspect of automatic structures, which has remained largely unexplored concerns the richness of the various automatic presentations a structure may possess. The few exceptions being numeration systems for $(\mathbb{N}, +)$ [3, 7], automatic presentations of $(\mathbb{Q}, <)$ [17], as well as of $(\mathbb{N}, <)$ and $(\mathbb{N}, S)$ [16]. The paper at hand presents contributions to this area.

Recently, Khoussainov et al. have introduced the notion of *intrinsic regularity* [16]. A relation is intrinsically regular wrt. a given structure, if it is mapped to a regular relation in *every* automatic presentation of that structure. Khoussainov et al. have shown that with respect to each automatic structure every relation definable in it using first-order logic with counting quantifiers is intrinsically regular. In [20, Problem 3] Rubin asked whether the converse of this is also true. In Section 3 we show that this is not the case. First, we observe that relations *order-invariantly* definable in first-order logic (with counting) are intrinsically regular with respect to each structure. Next, by adapting a technique of Otto [19] we exhibit an automatic structure together with an order-invariantly definable relation, which is not definable in any extension of first-order logic with *unary generalized quantifiers*. In [16, Question 1.4] Khoussainov et al. have called for a logical characterization of intrinsic regularity. Our example shows that it is not sufficient to add only unary generalized quantifiers to the language.

We propose to call two automatic presentations of the same structure equivalent, whenever they map exactly the same relations to regular ones. In Section 4 we investigate automatic presentations of $\mathfrak{S}_\Sigma$, where Σ is non-unary. Due to completeness, every a.p. of $\mathfrak{S}_\Sigma$ maps regular relations to regular ones. Our first result, Theorem 1, establishes, that, conversely, every a.p. of $\mathfrak{S}_\Sigma$ maps non-regular relations to non-regular ones. As a consequence we observe that $\mathfrak{S}_\Sigma$ has, up to equivalence, but one automatic presentation and that non-definable relations over $\mathfrak{S}_\Sigma$ are therefore intrinsically non-regular with respect to it.

Turning our attention to regularity-preserving mappings we introduce *semi-synchronous rational transducers*. These are essentially synchronized transducers in the sense of [10] with the relaxation that they may read each tape at a different, but constant pace. Our main result, Theorem 2 of Section 5, is that a bijection between regular languages preserves regularity of all *relations* in both directions if and only if it is a semi-synchronous transduction. It follows that two automatic presentations of an automatic structure are equivalent precisely when the bijection translating names of elements from one automatic presentation to the other is a semi-synchronous transduction.

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2 Preliminaries

Multi-tape automata Let Σ be a finite alphabet. The length of a word $w \in \Sigma^*$ is denoted by $|w|$, the empty word by ε, and for each $0 < i \leq |w|$ the ith symbol of w is written as $w[i]$. We consider relations on words, i.e. subsets R of $(\Sigma^*)^n$ for some $n > 0$. *Asynchronous* n-tape automata accept precisely the *rational relations*, i.e., rational subsets of the product monoid $(\Sigma^*)^n$. Finite *transducers*, recognizing *rational transductions* [2], are asynchronous 2-tape automata. A relation $R \subseteq (\Sigma^*)^n$ is *synchronized rational* [10] or *regular* [15] if it is accepted by a *synchronous* n-tape automaton. We introduce the following generalization.

Definition 1 (Semi-synchronous rational relations). *Let $\Box$ be a special end-marker symbol, $\Box \notin \Sigma$, and $\Sigma_\Box = \Sigma \cup \{\Box\}$. Let $\alpha = (a_1, \ldots, a_n)$ be a vector of positive integers and consider a relation $R \subseteq (\Sigma^*)^n$. Its* α-convolution *is $\boxtimes_\alpha R = \{(w_1\Box^{m_1}, \ldots, w_n\Box^{m_n}) \mid (w_1, \ldots, w_n) \in R$ and the m_i are minimal, such that there is a k, with $ka_i = |w_i| + m_i$ for every $i\}$. This allows us to identify $\boxtimes_\alpha R$ with a subset of the monoid $((\Sigma_\Box)^{a_1} \times \cdots \times (\Sigma_\Box)^{a_n})^*$. If $\boxtimes_\alpha R$ thus corresponds to a regular set, then we say that R is* α-synchronous (rational). *R is* semi-synchronous *if it is α-synchronous for some α.*

Intuitively, our definition expresses that although R requires an asynchronous automaton to accept it, synchronicity can be regained when processing words in blocks, the size of which are component-wise fixed by α. As a special case, for $\alpha = \mathbf{1}$, we obtain the regular relations. Recall that a relation $R \subseteq (\Sigma^*)^n$ is *recognizable* if it is saturated by a congruence (of the product monoid $(\Sigma^*)^n$) of finite index, equivalently, if it is a finite union of direct products of regular languages [10]. We denote by Rat, SRat, $\mathsf{S}_\alpha\mathsf{Rat}$, Reg, Rec the classes of rational, semi-synchronous, α-synchronous, regular, and recognizable relations respectively.

Automatic structures We take all structures to be relational with functions represented by their graphs.

Definition 2 (Automatic structures). *A structure $\mathfrak{A} = (A, \{R_i\}_i)$ consisting of relations R_i over the universe $\mathsf{dom}(\mathfrak{A}) = A$ is* automatic *if there is a finite alphabet Σ and an* injective *naming function $f : A \to \Sigma^*$ such that $f(A)$ is a regular subset of Σ^*, and the images of all relations of $\mathfrak{A}$ under f are in turn regular in the above sense. In this case we say that f is an* (injective) automatic presentation *of $\mathfrak{A}$. The class of all injective automatic presentations of $\mathfrak{A}$ is denoted $\mathsf{AP}(\mathfrak{A})$.* AutStr *designates the class of automatic structures.*

Example 1. Let Σ be a finite alphabet. Let S_a, $\preceq$ and el denote the a-successor relation, the prefix ordering, and the relation consisting of pairs of words of equal length. These relations are clearly regular, thus $\mathfrak{S}_\Sigma = (\Sigma^*, (S_a)_{a\in\Sigma}, \preceq, \mathsf{el})$ is automatic, having $\mathsf{id} \in \mathsf{AP}(\mathfrak{S}_\Sigma)$. Note that $\mathfrak{S}_{\{1\}}$ is essentially $(\mathbb{N}, \leq)$.

We use the abbreviation FO for first-order logic, and $\mathsf{FO}^{\infty,\mathrm{mod}}$ for its extension by infinity ($\exists^{\infty}$) and modulo counting quantifiers ($\exists^{(r,m)}$). The meaning of the formulae $\exists^{\infty} x\, \theta$ and $\exists^{(r,m)} x\, \theta$ is that there are infinitely many elements x, respectively r many elements x modulo m, such that θ holds. We shall make extensive use of the following facts, often without any reference.

Proposition 1. *(Consult [4, 6] and [16, 20].)*

i) Let $\mathfrak{A} \in$ AutStr, $f \in \mathsf{AP}(\mathfrak{A})$. Then one can effectively construct for each $\mathsf{FO}^{\infty,\mathrm{mod}}$-formula $\varphi(\boldsymbol{a}, \boldsymbol{x})$ with parameters $\boldsymbol{a}$ from $\mathfrak{A}$, defining a k-ary relation R over $\mathfrak{A}$, a k-tape synchronous automaton recognizing $f(R)$.

ii) The $\mathsf{FO}^{\infty,\mathrm{mod}}$-theory of any automatic structure is decidable.

iii) AutStr *is effectively closed under $\mathsf{FO}^{\infty,\mathrm{mod}}$-interpretations.*

Moreover, for each non-unary Σ, $\mathfrak{S}_{\Sigma}$ is *complete*, in the sense of item ii) below, for AutStr with respect to first-order interpretations.

Proposition 2. *[6] Let Σ be a finite,* non-unary *alphabet.*

i) A relation R over Σ^* is regular if and only if it is definable in $\mathfrak{S}_{\Sigma}$.

ii) A structure $\mathfrak{A}$ is automatic if and only if it is first-order interpretable in $\mathfrak{S}_{\Sigma}$.

Intrinsic regularity Let $\mathfrak{A} = (A, \{R_i\}_i) \in$ AutStr and $f \in \mathsf{AP}(\mathfrak{A})$. By definition f maps every relation R_i of $\mathfrak{A}$ to a regular one. The previous observations yield that f also maps all relations $\mathsf{FO}^{\infty,\mathrm{mod}}$-definable in $\mathfrak{A}$ to regular ones. Intrinsically regular relations of structures were introduced in [16]. We shall also be concerned with the dual notion of intrinsic non-regularity.

Definition 3 (Intrinsic regularity). *Let $\mathfrak{A}$ be automatic. The* intrinsically (non-)regular *relations of $\mathfrak{A}$ are those, which are (non-)regular in every a.p. of $\mathfrak{A}$. Formally,* $\mathsf{IR}(\mathfrak{A}) = \{R \subseteq A^r \mid r \in \mathbb{N}, \forall f \in \mathsf{AP}(\mathfrak{A})\, f(R) \in \mathsf{Reg}\}$ *and* $\mathsf{INR}(\mathfrak{A}) = \{R \subseteq A^r \mid r \in \mathbb{N}, \forall f \in \mathsf{AP}(\mathfrak{A})\, f(R) \notin \mathsf{Reg}\}$.

For any given logic $\mathcal{L}$ extending FO let $\mathcal{L}(\mathfrak{A})$ denote the set of relations over $\mathsf{dom}(\mathfrak{A})$ definable by an $\mathcal{L}$-formula using a finite number of parameters.

Remark 1. [16] $\mathsf{FO}^{\infty,\mathrm{mod}}(\mathfrak{A}) \subseteq \mathsf{IR}(\mathfrak{A})$ holds, by Prop. 1 i), for every $\mathfrak{A} \in$ AutStr.

Khoussainov et al. asked whether there is a logic $\mathcal{L}$ capturing intrinsic regularity, i.e., such that $\mathcal{L}(\mathfrak{A}) = \mathsf{IR}(\mathfrak{A})$ for all $\mathfrak{A} \in$ AutStr. We address this question in Section 3.

Example 2. Consider the structure $\mathcal{N} = (\mathbb{N}, +)$. For any integer $p \geq 2$ the base p (least-significant digit first) encoding provides an automatic presentation of $\mathcal{N}$. None of these presentations can be considered "canonical". On the contrary, by a deep result of Cobham and Semenov $\mathsf{base}_p^{-1}[\mathsf{Reg}] \cap \mathsf{base}_q^{-1}[\mathsf{Reg}] = \mathsf{FO}(\mathcal{N})$ for any p and q having no common power (cf. [3, 7]), hence $\mathsf{FO}(\mathcal{N}) = \mathsf{IR}(\mathcal{N})$.

When studying intrinsic regularity, it is natural to distinguish automatic presentations based on which relations they map to regular ones. To this end we introduce the following notion.

Definition 4 (Equivalence of automatic presentations).
For any $f, g \in \mathsf{AP}(\mathfrak{A})$ let $f \sim g \overset{\mathrm{def}}{\iff} f^{-1}[\mathsf{Reg}] = g^{-1}[\mathsf{Reg}]$.

Translations Let $\mathfrak{A} \in$ AUTSTR and $f, g \in \mathsf{AP}(\mathfrak{A})$. The mapping $t = g \circ f^{-1}$ is a bijection between regular languages, which *translates* names of elements of $\mathfrak{A}$ from one presentation to the other. Additionally, t preserves regularity of (the presentation of) all intrinsically regular relations of $\mathfrak{A}$.

Definition 5 (Translations). *A* translation *is a bijection $t : D \to C$ between regular sets $D \subseteq \Sigma^*$, $C \subseteq \Gamma^*$. If $D = \Sigma^*$ then t is a* total- *otherwise a* partial *translation. A translation t* preserves regularity (non-regularity) *if the image of every regular relation under t (respectively under t^{-1}) is again regular. Finally, t is* weakly regular *if it preserves both regularity and non-regularity.*

Note that by Proposition 2 all automatic presentations of $\mathfrak{S}_\Sigma$ are regularity preserving total translations. In general, one can fix a presentation $f \in \mathsf{AP}(\mathfrak{A})$ of every $\mathfrak{A} \in$ AUTSTR and consider instead of each presentation $g \in \mathsf{AP}(\mathfrak{A})$ the translation $t = g \circ f^{-1} \in \mathsf{AP}(f(\mathfrak{A}))$ according to the correspondence $\mathsf{AP}(\mathfrak{A}) = \mathsf{AP}(f(\mathfrak{A})) \circ f$. Also observe that $t = g \circ f^{-1}$ is weakly regular if and only if $f \sim g$. This holds, in particular, when t is a bijective synchronized rational transduction, referred to as an "automatic isomorphism" in [17] and [15]. Clearly, every bijective rational transduction qualifies as a translation, however, not necessarily weakly regular. In Section 5 we will show that weakly regular translations coincide with bijective semi-synchronous transductions.

We associate to each translation f its *growth function* G_f defined as $G_f(n) = max\{|f(u)| : u \in \Sigma^*, |u| \leq n\}$ for each n and say that f is *length-preserving* if $|f(x)| = |x|$ for every word x, further, f is *monotonic* if $|x| \leq |y|$ implies $|f(x)| \leq |f(y)|$ for every x and y, finally, f *has bounded delay* if there exists a constant δ such that $|x| + \delta < |y|$ implies $|f(x)| < |f(y)|$ for every x and y.

3 Order-invariant logic

In [16, Question 1.4] Khoussainov et al. have called for a logical characterization of intrinsic regularity over all automatic structures. The same question, and in particular, whether $\mathsf{FO}^{\infty,\mathrm{mod}}$ is capable of defining all intrinsically regular relations over any automatic structure is raised in [20, Problem 3]. In this section we answer the latter question negatively. We do this by exhibiting an automatic structure $\mathfrak{B}$ together with a relation, which is order-invariantly definable, but not $\mathsf{FO}^{\infty,\mathrm{mod}}$-definable in $\mathfrak{B}$.

Let $\mathfrak{A}$ be a structure of signature τ. Assume that $<$ is a binary relation symbol not occurring in τ. A formula $\phi(\boldsymbol{x}) \in \mathsf{FO}[\tau, <]$ is *order-invariant over* $\mathfrak{A}$ if for any linear ordering $<_A$ of the elements of $\mathfrak{A}$, when $<$ is interpreted as $<_A$, $\phi(\boldsymbol{x})$ defines the same relation R over $\mathfrak{A}$. The relation R is in this case *order-invariantly definable.* We denote the set of order-invariantly definable relations over $\mathfrak{A}$ by $\mathsf{FO}_{<-\mathsf{inv}}(\mathfrak{A})$, and by $\mathsf{FO}^{\infty,\mathrm{mod}}_{<-\mathsf{inv}}(\mathfrak{A})$ when counting quantifiers are also allowed. Although it is only appropriate to speak of order-invariantly definable relations, rather than of relations definable in "order-invariant logic", we will tacitly use the latter term as well.

The fact that over any $\mathfrak{A} \in$ AutStr order-invariantly definable relations are intrinsically regular is obvious. Indeed, given a particular automatic presentation of $\mathfrak{A}$ one just has to "plug in" any regular ordering (e.g. the lexicographic ordering, which does of course depend on the automatic presentation chosen) into the order-invariant formula defining a particular relation, thereby yielding a regular relation, which, by order-invariance, will always represent the same relation.

Proposition 3. $\mathsf{FO}^{\infty,\mathrm{mod}}_{<-\mathsf{inv}}(\mathfrak{A}) \subseteq \mathsf{IR}(\mathfrak{A})$

Order-invariant first-order logic has played an important role in finite model theory. It is well known that $\mathsf{FO}_{<-\mathsf{inv}}$ is strictly more expressive than FO on finite structures. Gurevich was the first to exhibit an order-invariantly definable class of finite structures, which is not first-order definable [18, Sect. 5.2]. However, his class is $\mathsf{FO}^{\infty,\mathrm{mod}}$-definable. In [19] Otto showed how to use order-invariance to express connectivity, which is not definable even in infinitary counting logic. Both constructions use order-invariance and some auxiliary structure to exploit the power of monadic second order logic (MSO). We adopt Otto's technique to show that $\mathsf{FO}_{<-\mathsf{inv}}$ can be strictly more expressive than $\mathsf{FO}^{\infty,\mathrm{mod}}$ on automatic structures. The proof involves a version of the bijective Ehrenfeucht-Fraïssé games, introduced by Hella [11], which capture equivalence modulo $\mathsf{FO}(\mathbf{Q}_1)$, the extension of FO with *unary generalized quantifiers* [18, Chapter 8]. The simples ones of these are $\exists^\infty$ and $\exists^{(r,m)}$. Therefore, as also observed by Otto, the separation result applies not only to $\mathsf{FO}^{\infty,\mathrm{mod}}$ but to the much more powerful logic $\mathsf{FO}(\mathbf{Q}_1)$.

Consider the structure $\mathfrak{B} = (\mathbb{N} \uplus \mathcal{P}_{\mathrm{fin}}(4\mathbb{N} + \{2,3\}), S, \varepsilon, \iota, \subseteq)$, illustrated in Figure 1, where $\mathcal{P}_{\mathrm{fin}}(H)$ consists of the finite subsets of H, S is the relation $\{(4n, 4n+4), (4n+1, 4n+5) \mid n \in \mathbb{N}\}$, ε is the equivalence relation consisting of classes $\{4n, 4n+1, 4n+2, 4n+3\}$ for each $n \in \mathbb{N}$, ι is the set of pairs $(n, \{n\})$ with $n \in 4\mathbb{N} + \{2,3\}$, and $\subseteq$ is the usual subset inclusion.

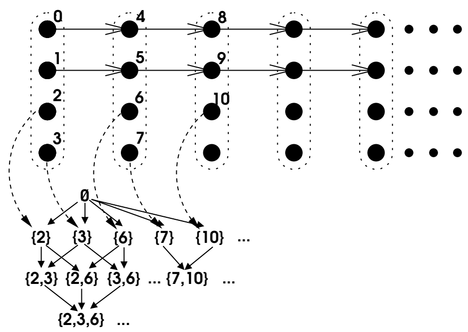

Fig. 1. $\mathfrak{B}$, a separating example.

To give an automatic presentation of $\mathfrak{B}$ over the alphabet $\{b, 0, 1\}$ we represent $(\mathbb{N}, S, \varepsilon)$ in the unary encoding using the symbol b, and the finite sets by their (shortest) characteristic words over $\{0,1\}$. Regularity of ι and $\subseteq$ is obvious.

Proposition 4. *The transitive closure S^* of S is order-invariantly definable, hence intrinsically regular, but not* $\mathsf{FO}(\mathbf{Q}_1)$*-definable in* $\mathfrak{B}$.

Proof. The proof is a straightforward adaptation of the one presented in [19].
$S^* \in \mathsf{FO}_{<-\mathsf{inv}}(\mathfrak{B})$: Given any ordering $\prec$ of the universe of $\mathfrak{B}$ we can first-order define a bijection $\nu = \nu_\prec : 4\mathbb{N} \cup 4\mathbb{N}+1 \to 4\mathbb{N}+2 \cup 4\mathbb{N}+3$ as follows. Each ε-class contains two points, $4n+2$ and $4n+3$, having an outgoing ι edge and two points, $4n$ and $4n+1$, having an S-successor. Using $\prec$ we can map e.g. the smaller (larger) of the latter to the smaller (larger) of the former. This bijection, regardless of the actual mapping, provides access to the subset structure, thus unleashing full power of weak-MSO. Transform, using ν and the built-in subset structure, any weak-MSO definition of transitive closure into one expressing S^*.
$S^* \notin \mathsf{FO}(\mathbf{Q}_1)(\mathfrak{B})$: The proof of this statement involves a fairly standard application of bijective Ehrenfeucht-Fraïssé games as hinted in [19]. □

Corollary 1. *No extension of* FO *with unary generalized quantifiers is capable of capturing intrinsic regularity over all automatic structures.*

4 Automatic presentations of $\mathfrak{S}_\Sigma$

Recall $\mathfrak{S}_\Sigma$ of Example 1. Let Σ be non-unary. The main result of this section is that automatic presentations of $\mathfrak{S}_\Sigma$ are weakly regular translations, hence are all equivalent. We treat the case of length-preserving presentations first.

Proposition 5. *Let $f : \Sigma^* \to \Gamma^*$ be a length-preserving automatic presentation of $\mathfrak{S}_\Sigma$. Then (the graph of) f is regular.*

Proof. Consider $\{(u,v) \in (\Sigma^*)^2 \,:\, |u| \leq |v| \wedge v[|u|] = a\}$, which is clearly regular for each $a \in \Sigma$. Their images under f are regular relations over Γ^*. Since only the length of the first component plays a role in these relations, and it is preserved by f, the following "variants" over $\Sigma^* \times \Gamma^*$ are also regular.

$$R_a = \{(u,x) \in \Sigma^* \times f(\Sigma^*) \,:\, |u| \leq |x| \wedge f^{-1}(x)[|u|] = a\} \quad (a \in \Sigma)$$

Thus, we may define the graph of f as follows showing that it is indeed regular.

$$\mathrm{graph}(f) = \{(u,x) \in \Sigma^* \times \Gamma^* \,:\, |u| = |x| \wedge \forall v \preceq u \bigwedge_{a\in\Sigma} v[|v|] = a \to R_a(v,x)\} \ \square$$

Theorem 1. $f \in \mathsf{AP}(\mathfrak{S}_\Sigma) \iff$ *f is a total and weakly regular translation.*

Proof. We only need to prove "⇒". Let $f \in \mathsf{AP}(\mathfrak{S}_\Sigma)$. In two steps of transformations we will show that f is equivalent to a length-preserving presentation h of $\mathfrak{S}_\Sigma$. The claim then follows by Proposition 5.

The relations $|y| \leq |x|$, and S_a $(a \in \Sigma)$ are locally finite and regular. A standard pumping argument (e.g. [14, Lemma 3.1]) then shows, that there are constants K and C such that $|y| \leq |x| \to |f(y)| \leq |f(x)| + K$ and $|f(xa)| \leq |f(x)| + C$ for every $a \in \Sigma$ and $x, y \in \Sigma^*$. It is easily seen, that by suffixing each codeword $f(x)$ by an appropriate ($\leq K$) number of some new padding symbols, we can obtain an equivalent monotonic presentation g.

Lemma 1. $\forall f \in \mathsf{AP}(\mathfrak{S}_\Sigma)\ \exists g \in \mathsf{AP}(\mathfrak{S}_\Sigma) : g \sim f,\ g$ *is monotonic and* $G_g = G_f$.

Proof. By the choice of K above, we have $G_f(|x|) \leq |f(x)| + K$, and for each $s = 0..K$ the set $D_s = \{x : G_f(|x|) - |f(x)| = s\}$ is regular, being definable. This observation allows us to pad each codeword accordingly. Let us therefore define g by letting $g(x) = f(x)@^{G_f(|x|)-|f(x)|}$ for every $(x \in \Sigma^*)$. where @ is a new padding symbol. The domain of the new presentation, that is $g(\Sigma^*) = \bigcup_{s=1}^{k} f(D_s) \cdot @^s$ is by the above argument regular. Moreover, since this padding is definable f and g map the same relations to regular ones. Finally, it is clear that g is monotonic, because $|g(x)| = G_f(|x|) = G_g(|x|)$ holds for every word x, and the growth function G_f is by definition always monotonic. □

The decisive step of the construction requires two key lemmas.

Lemma 2. $\forall f \in \mathsf{AP}(\mathfrak{S}_\Sigma)$ *: f has bounded delay.*

Proof. Consider the equivalent presentation g obtained from f by padding each codeword with at most K new symbols as in Lemma 1. If g has bounded delay with bound δ then f has bounded delay with bound $\leq K\delta$. Assume therefore that f is monotonic. Let $D = f(\Sigma^*)$, $s = |\Sigma| \geq 2$, and $D_{\leq n} = \{x \in D \mid |x| \leq n\}$ for each $n \in \mathbb{N}$. Assume, that for some n and t we find the following situation.

$$G_f(n-1) < G_f(n) = G_f(n+1) = \ldots = G_f(n+t-1) < G_f(n+t)$$

Then $|D_{\leq G_f(n-1)}| = (s^n - 1)/(s-1)$ and $|D_{\leq G_f(n)}| = |D_{\leq G_f(n+t-1)}| = (s^{n+t} - 1)/(s-1)$ since they contain (due to monotonicity) precisely the images of words of length at most $n-1$ and $n+t-1$ respectively. On the other hand, by the choice of C, we have $G_f(n) \leq G_f(n-1) + C$, hence $D_{\leq G_f(n)} \subseteq D_{\leq G_f(n-1)+C}$ for every $n \in \mathbb{N}$. In [14, Lemma 3.12] it is shown that $|D_{\leq n+C}| \in \Theta(|D_{\leq n}|)$ for each C. Thus, there is a constant β (certainly, $\beta \geq 1$) such that $|D_{\leq G_f(n)}| \leq |D_{\leq G_f(n-1)+C}| \leq \beta \cdot |D_{\leq G_f(n-1)}|$. By simple arithmetic, $t \leq log_s(\beta)$, which proves that f has bounded delay, namely, bounded by $\delta = log_s(\beta) + 1$. □

Lemma 3. *For all $f \in \mathsf{AP}(\mathfrak{S}_\Sigma)$ the infinite sequence of increments of the growth function of f, $\partial G_f = \langle G_f(1) - G_f(0), \ldots, G_f(n+1) - G_f(n), \ldots\rangle \in \{0, \ldots, C\}^\omega$, is ultimately periodic.*

Proof. Consider the monotonic mapping g obtained from f by padding as in Lemma 1, and the language $L = \{x = g(u) \mid \forall y = g(v)(|u| = |v| \to x \leq_{llex} y)\}$ consisting of the length-lexicographically least g-representants of some word of length n for each non-negative n. L is regular and since g has bounded delay, say with bound δ, it is δ-thin, meaning that there are at most δ many words in L of each length. We can thus write L as disjoint union of the regular languages $L_k = \{x \in L \mid \exists^{=k} y \in L\ |x| = |y|\}$ for $k = 1, \ldots, \delta$. Let us project L as well as L_k's onto 1^* in length-preserving manner. $G_g = G_f$ is a non-decreasing sequence of naturals in which each number can occur at most δ times. The projection of L corresponds, in the unary encoding, to the pruned sequence obtained from G_f by omitting the repetitions, whereas L_k is mapped onto those 1^n for which

n is repeated exactly k times in G_f. All these projections are regular unary languages, which is the same as saying that the corresponding sets of naturals are ultimately periodic. The claim follows. □

This last result allows us to construct an equivalent length-preserving a.p. h by factoring each word $g(u)$ of length $G_g(|u|)$ into "blocks" according to ∂G_g.

Lemma 4. *$\forall g \in \mathsf{AP}(\mathfrak{S}_\Sigma) : g$ is monotonic $\rightarrow \exists h \sim g : h$ is length-preserving.*

Proof. Let $g : \Sigma^* \rightarrow \Gamma^*$ be a monotonic a.p. of $\mathsf{AP}(\mathfrak{S}_\Sigma)$ with $D = g(\Sigma^*)$. The fact that ∂G_g is ultimately periodic allows us to construct an equivalent length-preserving presentation h by subdividing codewords produced by g into blocks according to ∂G_g. (For this we need to assume that $G_g(0) = 0$, i.e. the empty word is represented by itself. Clearly, this is no serious restriction as changing an a.p. on a finite number of words always yields an equivalent a.p.)

Consider some word $u \in \Sigma^*$ of size n and its image $v = g(u) \in \Gamma^*$. Since g is monotonic $|v| = G_g(|u|) = G_g(n)$. Thus we can factorize v as $v_1 v_2 \cdots v_n$ where $|v_i| = \partial G_g[i]$ for each $i \leq n$. Since $\partial G_g[i] \leq C$ for every i, we can consider each v_i as a single symbol of the alphabet $\Theta = \Gamma^{\leq C} = \{w \in \Gamma^* : |w| \leq C\}$. Let β be the natural projection mapping elements of Θ to the corresponding words over Γ, and let $\lambda(w) = |\beta(w)|$ for each $w \in \Theta$.

We define the mapping $h : \Sigma^* \rightarrow \Theta^*$ by setting for each $u \in \Sigma^*$, with factorization as above, $h(u) = v_1 \cdot v_2 \cdot \ldots \cdot v_n$ when considered as a word of length n over Θ. Thus, h is by definition length-preserving and maps Σ^* injectively onto the set $D' = \{x \in \Theta^* \,|\, \beta(x) \in D \wedge (\forall i = 1..|x|)\ \lambda(x[i]) = \partial G_g(i)\}$. Because β is a homomorphism, D regular and ∂G_g ultimately periodic, D' can clearly be accepted by a finite automaton. Moreover, the fact that any two words w,w' belonging to D' are synchronously blocked (in the sense that $x[i]$ and $x'[i]$ have the same length for all $i \leq |x|, |x'|$) enables us to easily simulate any n-tape automaton $\mathcal{A}$ accepting a relation over D by an automaton $\mathcal{A}'$ accepting the "same" relation over D' and vice versa. □

This concludes the proof of Theorem 1. □

Corollary 2. *Non-regular relations are intrinsically non-regular wrt. $\mathfrak{S}_\Sigma$.*

Corollary 3. *Every total translation that preserves regularity also preserves non-regularity, hence is weakly regular.*

Theorem 1 fails for unary alphabets, because, as can easily be checked, the mapping from unary to binary presentation of the naturals does preserve regularity, but also maps some non-regular relations to regular ones. The same argument shows that Corollary 3 does not hold for partial translations: simply take a "variant" of the unary presentation over the partial domain $(ab)^* \subsetneq \{a, b\}^*$.

Corollary 4. *The complete structures $\mathfrak{S}_\Sigma$ have, up to equivalence, only a single automatic presentation:* $\mathsf{AP}(\mathfrak{S}_\Sigma)/_\sim = \{[\mathrm{id}]\}$

Not all complete structures have this property. Let $\mathfrak{C} = \mathfrak{A} \uplus \mathfrak{B}$ be the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$ having an additional unary predicate A identifying elements belonging to $\mathfrak{A}$. Thus, $\mathfrak{A}$ and $\mathfrak{B}$ are trivially FO-interpretable in $\mathfrak{C}$, and $\mathfrak{C} \in$ AutStr iff $\mathfrak{A}, \mathfrak{B} \in$ AutStr. It follows from Proposition 7 below, that $\mathfrak{A} \uplus \mathfrak{B}$ has infinitely many inequivalent automatic presentations, provided both $\mathfrak{A}$ and $\mathfrak{B}$ are infinite. In particular, this holds for the complete structure $\mathfrak{S}_\Sigma \uplus \mathfrak{S}_\Sigma$. Let us therefore say that a structure is *rigidly automatic* if it has but one automatic presentation up to equivalence. Finite structures are trivially rigidly automatic.

Question 1. Is every infinite, rigidly automatic structure complete?

5 Equivalence via semi-synchronous transductions

Observe that we have proved more than what is claimed in Theorem 1. The above proof shows indeed, that every $f \in \mathsf{AP}(\mathfrak{S}_\Sigma)$ can be decomposed as

$$f = \pi^{-1} \circ \beta^{-1} \circ h$$

where π applies the padding, β the cutting of words into blocks, and where h is length-preserving and regular. Since both π^{-1} and β^{-1} are projections the composition is a rational transduction.[1] Moreover, we know that ∂G_f is ultimately periodic, say from threshold N with period p. Let $q = G_f(N+p) - G_f(N)$ be the total length of *any* p consecutive blocks with sufficiently high indices. This means that after reading the first N input symbols and the first $G_f(N)$ output symbols a transducer accepting f can proceed by reading blocks of p input- and q output symbols in each step, which shows that f is in fact a (p,q)-synchronous transduction. This decomposition is the idea underlying one direction of the main result of this section, the next lemma constitutes the other.

Lemma 5. *For every vector α of nonnegative integers $\mathsf{S}_\alpha\mathsf{Rat}$ is closed under taking images (hence also inverse images) of semi-synchronous transductions.*

Proof. Let T be a (p,q)-synchronous transduction, R an α-synchronous n-ary relation with $\alpha = (a_1, \ldots, a_n)$. $T(R) = \{\boldsymbol{v} \mid \exists \boldsymbol{u} \in R\ \forall i \leq n\ (u_i, v_i) \in T\}$ is the projection of the $(pa_1, \ldots, pa_n, qa_1, \ldots, qa_n)$-synchronous relation $\{(\boldsymbol{u}, \boldsymbol{v}) \mid \boldsymbol{u} \in R\ \forall i \leq n\ (u_i, v_i) \in T\}$. Hence, by Propositions 6 and 7, $T(R)$ is α-synchronous. Closure under taking inverse images follows from the fact, that the inverse of a (p,q)-synchronous transduction is (q,p)-synchronous. □

Theorem 2. *A translation f is weakly regular if and only if it is a semi-synchronous transduction.*

Proof. The "if" part is a special case of Lemma 5. To prove the "only if" part we show that, under the assumption of weak-regularity, all lemmas used to prove

[1] Knowing this, Proposition 5 follows from [9, Corollary 6.6] stating that length-preserving transductions are synchronized rational. (See also [10].)

Theorem 1 hold even for partial translations. We only note that lemmas 1, 3 and 4 carry over without modification. To prove that a monotonic, possibly partial, weakly-regular translation g has bounded delay it suffices to consider the inverse image of the predicate $|x| \leq |y|$ under g. A pumping argument shows that there is a constant D such that $|g^{-1}(x)| \leq |g^{-1}(y)| + D$ whenever $|x| \leq |y|$, in other words $|u| > |v| + D$ implies that $|g(u)| > |g(v)|$, i. e. g has bounded delay. This proves the analog of Lemma 2.[2] Finally, note that our proof of Proposition 5 works only assuming that the domain (hence the range) of the length-preserving weakly regular mapping considered contains at least one word of every length. This requirement is not essential. Instead of R_a's ($a \in \Sigma$) one can just as well use the relations R_w defined for each $w \in \Sigma^k$ in the obvious way for a sufficiently large k. Thus, we obtain the same decomposition $f = \pi^{-1} \circ \beta^{-1} \circ h$, which shows, as above, that f is a semi-synchronous transduction. □

Corollary 5. *Let $\mathfrak{A} \in$ AutStr and $f, g \in AP(\mathfrak{A})$. Then $f \sim g$ if and only if the translation $g \circ f^{-1}$ (and $f \circ g^{-1}$) is a semi-synchronous transduction.*

Basic properties of semi-synchronous relations Note that the composition of a (p, q)-synchronous and an (r, s)-synchronous transduction is (pr, qs)-synchronous, thus, the class of semi-synchronous transductions is closed under composition. Alternative to our definition of $\mathsf{S}_\alpha\mathsf{Rat}$ based on α-convolution one can introduce *α-synchronous automata*, defined in the obvious way, accepting α-synchronous relations. These automata, being essentially synchronous, can be determinized, taken product of, etc. Hence the following.

Proposition 6. *$\mathsf{S}_\alpha\mathsf{Rat}$ is an effective boolean algebra for each α. The projection of every $\alpha\beta$-synchronous relation onto the first $|\alpha|$ many components, is α-synchronous.*

Evidently, $\mathsf{Reg} \subset \mathsf{SRat} \subset \mathsf{Rat}$ and both containments are strict as illustrated by the examples $\{(a^n, a^{2n}) \mid n \in \mathbb{N}\}$ and $\{(a^n, a^{2n}), (b^n, b^{3n}) \mid n \in \mathbb{N}\}$. SRat is closed under complement but not under union, as also shown by the latter example. Comparing classes $\mathsf{S}_\alpha\mathsf{Rat}$ and $\mathsf{S}_\beta\mathsf{Rat}$ we observe the following "Cobham-Semenov-like" relationship. Let us say that α and β are dependent if $k \cdot \alpha = l \cdot \beta$ for some $k, l \in \mathbb{N}$, where multiplication is meant component-wise.

Proposition 7. *Let $n, p, q \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}^n$.*
i) If α and β are dependent, then $\mathsf{S}_\alpha\mathsf{Rat} = \mathsf{S}_\beta\mathsf{Rat}$.
ii) If (p, q) and (r, s) are independent, then $\mathsf{S}_{(p,q)}\mathsf{Rat} \bigcap \mathsf{S}_{(r,s)}\mathsf{Rat} = \mathsf{Rec}$.

Adapting techniques from [2, 10], used to prove undecidability of whether a given rational relation is synchronized rational, we obtain the following results.

Proposition 8. *For any given $p, q \in \mathbb{N}$ the following problems are undecidable.*
i) Given a rational transduction $R \in \mathsf{Rat}$ is $R \in \mathsf{S}_{(p,q)}\mathsf{Rat}$?
ii) Given a rational transduction $R \in \mathsf{Rat}$ is $R \in \mathsf{SRat}$?

[2] Note that this argument was not applicable in the proof of Lemma 2, since at that point we could not rely on the mapping preserving non-regularity but only on it being total.

Future work

Themes of follow-up work include generalizing results to ω-automatic structures, questions concerning the number of automatic presentations modulo equivalence, and a finer analysis of definability of intrinsically regular relations on restricted classes of structures.

References

1. J.-P. Allouche and J. Shallit. *Automatic Sequences, Theory, Applications, Generalizations.* Cambridge University Press, 2003.
2. J. Berstel. *Transductions and Context-Free Languages.* Teubner, Stuttgart, 1979.
3. A. Bès. Undecidable extensions of Büchi arithmetic and Cobham-Semënov theorem. *J. Symb. Log.*, 62(4):1280–1296, 1997.
4. A. Blumensath. Automatic structures. Diploma thesis, RWTH-Aachen, 1999.
5. A. Blumensath and E. Grädel. Automatic Structures. In *Proceedings of 15th IEEE Symposium on Logic in Computer Science LICS 2000*, pages 51–62, 2000.
6. A. Blumensath and E. Grädel. Finite presentations of infinite structures: Automata and interpretations. *Theory of Computing Systems*, 37:641 – 674, 2004.
7. V. Bruyère, G. Hansel, Ch. Michaux, and R. Villemaire. Logic and p-recognizable sets of integers. *Bull. Belg. Math. Soc.*, 1:191 – 238, 1994.
8. J.W. Cannon, D.B.A. Epstein, D.F. Holt, S.V.F. Levy, M.S. Paterson, and W.P. Thurston. *Word processing in groups.* Jones and Barlett Publ., Boston, MA, 1992.
9. C. C. Elgot and J. E. Mezei. On relations defined by generalized finite automata. *IBM J. Research and Development*, 9:47 – 68, 1965.
10. Ch. Frougny and J. Sakarovitch. Synchronized rational relations of finite and infinite words. *Theor. Comput. Sci.*, 108:45–82, 1993.
11. L. Hella. Definability hierarchies of generalized quantifiers. *Ann. Pure Appl. Logic*, 43:235 – 271, 1989.
12. B.R. Hodgson. On direct products of automaton decidable theories. *TCS*, 19:331–335, 1982.
13. B. Khoussainov and A. Nerode. Automatic presentations of structures. In *LCC '94*, volume 960 of *LNCS*, pages 367–392. Springer-Verlag, 1995.
14. B. Khoussainov, A. Nies, S. Rubin, and F. Stephan. Automatic structures: Richness and limitations. In *LICS*, pages 44–53. IEEE Computer Society, 2004.
15. B. Khoussainov and S. Rubin. Automatic structures: Overview and future directions. *Journal of Automata, Languages and Combinatorics*, 8(2):287–301, 2003.
16. B. Khoussainov, S. Rubin, and F. Stephan. Definability and regularity in automatic structures. In *STACS '04*, volume 2996 of *LNCS*, pages 440–451, 2004.
17. D. Kuske. Is cantor's theorem automatic? In M. Y. Vardi and A. Voronkov, editors, *LPAR*, volume 2850 of *LNCS*, pages 332–345. Springer, 2003.
18. L. Libkin. *Elements of Finite Model Theory.* Texts in Theoretical Computer Science. Springer, 2004.
19. M. Otto. Epsilon-logic is more expressive than first-order logic over finite structures. *J. Symb. Logic*, 65:1749 – 1757, 2000.
20. S. Rubin. Automatic structures. Ph.D. thesis, University of Auckland, NZ, 2004.

Appendix

A Details for Section 3

First we briefly recall the notions and results of [11], which we will be using. $\mathsf{FO}(\mathbf{Q}_1)$ is a proper syntactic extension of FO, defined by adding *unary generalized quantifiers* to the logic. A unary generalized quantifier is defined in terms of a class $\mathcal{K}$ of structures consisting of a universe and a fixed (possibly infinite) number of unary relations. Additionally we require $\mathcal{K}$ to be closed under isomorphisms, and associate a quantifier $\mathcal{Q}_\mathcal{K}$ to this class. The logic $\mathsf{FO}(\mathbf{Q}_1)$ consists of formulas in which, in addition to classical first-order constructs, quantifiers of the above kind are also allowed. Without giving a precise definition of the syntax we mention that the intended meaning of a formula $\mathcal{Q}_\mathcal{K}[\phi_1(x_1, \boldsymbol{z}), \ldots, \phi_n(x_n, \boldsymbol{z}), \ldots]$ over a structure $\mathfrak{A}$ with universe A is that for a given value $\boldsymbol{a}$ of the variables $\boldsymbol{z}$ the structure $(A, \phi_1^\mathfrak{A}(x, \boldsymbol{a}), \ldots, \phi_n^\mathfrak{A}(x, \boldsymbol{a}), \ldots)$ belongs to the class $\mathcal{K}$ where $\phi_i^\mathfrak{A}(x, \boldsymbol{a}) = \{c \in A \mid \mathfrak{A} \models \phi_i(c, \boldsymbol{a})\}$ for each i . Observe that the first-order quantifiers $\exists$, $\forall$ as well as the counting quantifiers $\exists^{r,m}$ and $\exists^\infty$ introduced above are particular unary generalized quantifiers.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures sharing a common signature. The *r-round bijective Ehrenfeucht-Fraïssé game* $\mathrm{BEF}_r(\mathfrak{A}, \mathfrak{B})$ is defined as follows. There are two players: I and II. The positions of the game are partial isomorphisms between the two structures, provided there are any, the initial position being the empty isomorphism. In case $\emptyset$ is not a partial isomorphism, the game is won by I up front without any moves having been made. In each round of the game, in position p, player II proposes a bijection $f : A \to B$ such that $p \cup (a, f(a))$ is again a partial isomorphism for every $a \in \mathsf{dom}(\mathfrak{A})$, or loses. Player I replies by choosing an element $a \in \mathsf{dom}(\mathfrak{A})$, thus determining the new position as $p \cup (a, f(a))$ (that is to say II fixed her reply $f(a)$ in advance). The game ends after at most r rounds. Player II wins if she does not lose in the mean time.

A strategy of player II in this game is captured by an *r-bijective back-and-forth system* consisting of a sequence $(I_i)_{i \le r}$ of sets of partial isomorphisms between $\mathfrak{A}$ and $\mathfrak{B}$, such that $\emptyset \in I_r$ and for every $k < r$ and $p \in I_{k+1}$ there is a bijection $f_p : A \to B$ for which $p \cup \{(a, f(a))\} \in I_k$ for every $a \in A$.

As shown in [11] two structures $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of $\mathsf{FO}(\mathbf{Q}_1)$ of quantifier rank at most r iff player II has a winning strategy in the game $\mathrm{BEF}_r(\mathfrak{A}, \mathfrak{B})$ iff there is an r-bijective back-and-forth system $(I_i)_{i \le r} : \mathfrak{A} \sim^r \mathfrak{B}$.

Proposition 4. *The transitive closure S^* of S is order-invariantly definable, hence intrinsically regular, but not* $\mathsf{FO}(\mathbf{Q}_1)$*-definable in* $\mathfrak{B}$*.*

Proof. The proof is an adaptation of the one presented in [19].
$S^* \in \mathsf{FO}_{<-\mathsf{inv}}(\mathfrak{B})$: Given any ordering $\prec$ of the universe of $\mathfrak{B}$ we can first-order define a bijection $\nu = \nu_\prec : 4\mathbb{N} \cup 4\mathbb{N}+1 \to 4\mathbb{N}+2 \cup 4\mathbb{N}+3$ as follows. Each ε-class contains two isolated points $4n+2$ and $4n+3$ and two points $4n$ and $4n+1$ having an S-successor for some n. Using $\prec$ we can map e.g. the smaller (larger) of the latter to the smaller (larger) of the former. This bijection, regardless of the actual mapping, provides access to the subset structure, thereby unleashing full wMSO power. Take any wMSO formula defining transitive closure and translate it using ν and the built-in subset structure to express S^*.
$S^* \notin \mathsf{FO}(\mathbf{Q}_1)(\mathfrak{B})$: Let $\mathfrak{B}_n = (\mathfrak{B}, 0, 4n)$ and $\mathfrak{B}'_n = (\mathfrak{B}, 0, 4n+1)$. It is sufficient to show that for large enough n player II wins $\mathrm{BEF}_r(\mathfrak{B}_n, \mathfrak{B}'_n)$.

Let $B = \mathsf{dom}(\mathfrak{B})$ and $D = \mathsf{dom}(S)$. Considering the reducts $\mathfrak{S}_n = \mathfrak{B}_n|_D$ and $\mathfrak{S}'_n = \mathfrak{B}'_n|_D$ it should be clear that player II has a winning strategy in the r-round bijective game $BEF_r(\mathfrak{S}_n, \mathfrak{S}'_n)$ for some $n \in 2^{O(r)}$. Moreover, there is an r-bijective back-and-forth system $(I_i)_{i \leq r} : \mathfrak{S}_n \sim^r \mathfrak{S}'_n$, such that for every $k \leq r$ each $p \in I_k$ maps $\{4m, 4m+1\}$ into itself for every m (*), i.e. ε-classes are preserved throughout any play consistent with this strategy.

We extend this strategy to one in $\mathrm{BEF}_r(\mathfrak{B}_n, \mathfrak{B}'_n)$ by extending the bijections given by the former strategy identically onto all elements outside of the domain of S. Equivalently, we claim that $(J_i)_{i \leq r} : \mathfrak{B}_n \sim^r \mathfrak{B}'_n$, where $J_k = \{p \cup q \mid p \in I_k, q \subset \mathsf{id}|_{B \setminus D}\}$ for each $k \leq r$. Each such $p \cup q$ is indeed a partial isomorphism, because both p and q are on the respective "halves" of the structures and p also satisfies (*). Further, for any $p \cup q \in J_{k+1}$, thus $p \in I_{k+1}$, there is by definition a bijection $f_p : D \to D$ such that $p \cup (a, f_p(a))$ is in I_k for any $a \in D$. Hence, with $g_p = f_p \cup \mathsf{id}|_{B \setminus D}$ it holds that $p \cup q \cup (a, g_p(a)) \in J_k$ for any $a \in B$. This concludes the proof. ☐

B Details for Section 5

Let us first recall some basic combinatorial facts, which can be found e.g. in [14]. The first one is a straightforward consequence of the well-known "pumping lemma" of automata theory. A relation R of arity $n+m$ is *locally finite* if for every $(x_1, \ldots, x_n)$ there are only finitely many $(y_1, \ldots, y_m)$ such that $R(\boldsymbol{x}, \boldsymbol{y})$ holds.

Proposition B1 *([4], [14]) Let $R \subseteq (\Sigma^*)^{n+m}$ be a regular and locally finite relation. Then there is a constant k such that $max_j|y_j| \leq max_i|x_i| + k$ holds for every $R(\boldsymbol{x}, \boldsymbol{y})$. In particular, if f is a regular function (e.g. S_a), then there is a constant k such that for every $\boldsymbol{x}$ in its domain we have $|f(\boldsymbol{x})| \leq max_i|x_i| + k$.*

Some examples of locally finite relations are el, $|x| > |y|$ and $y \preceq x$. Note that $x \preceq y$ is not locally finite in the above notation.

We shall take advantage of another combinatorial lemma that first appeared in [14]. For any regular set $D \subseteq \Sigma^*$ let $D_{=n} = D \cap \Sigma^n$ and $D_{\leq n} = D \cap \Sigma^{\leq n}$ denote the set of members of D of length precisely n and at most n respectively. Further let $\mathsf{Pref}(D)$ be the (regular) set of prefixes of words in D.

Proposition B2 *[14, Lemma 3.12] Let $D \subseteq \Sigma^*$ be a regular set. Then*

1. $|\mathsf{Pref}(D)_{=n}| = \mathcal{O}(|D_{\leq n}|)$ *and*
2. *for every fixed* $C \in \mathbb{N}$: $|D_{\leq(n+C)}| = \Theta(|D_{\leq n}|)$

Let us say that α and β are dependent if there are integers k and l such that $k\alpha = l\beta$, where multiplication is meant componentwise.
Proposition 7. *Let $n, p, q \in \mathbb{N}$ and $\alpha,\beta \in \mathbb{N}^n$.*

(1) If α and β are dependent, then $\mathsf{S}_\alpha\mathsf{Rat} = \mathsf{S}_\beta\mathsf{Rat}$.
(2) If (p, q) and (r, s) are independent, then $\mathsf{S}_{(p,q)}\mathsf{Rat} \bigcap \mathsf{S}_{(r,s)}\mathsf{Rat} = \mathsf{Rec}$.

Proof. (1) Clearly, a relation R is α-synchronous if and only if it is $k\alpha$-synchronous for any $k \geq 1$. The claim follows.
(2) Recognizable relations are trivially α-synchronous for any α, therefore we only care for the other inclusion.

Let $R \in \mathsf{S}_{(p,q)}\mathsf{Rat} \bigcap \mathsf{S}_{(r,s)}\mathsf{Rat}$. We need to show, that R is a finite union of Cartesian products $A_i \times B_i$ of regular languages, in other terms that the following equivalence is of finite index.

$$x \sim x' \overset{\text{def}}{\iff} \forall y : R(x, y) \leftrightarrow R(x', y)$$

According to (1) R is both (pr, qr)- and (pr, ps)-synchronous, and by assumption $ps \neq qr$, w.l.o.g. $ps < qr$. Let us further assume for simplicity and w.l.o.g. that $pr = 1$ and let $k = ps$ and $l = qr$. Consider some $(1, k)$- respectively $(1, l)$-synchronous deterministic automata $\mathcal{A}$ and $\mathcal{A}'$ accepting R. Thus $\mathcal{A}$ is "slower" then $\mathcal{A}'$ in reading the second tape. Our first observation is confirmed by a straightforward pumping argument.

$$x \not\sim x' \Rightarrow \exists y : |y| < k(\max(|x|, |x'|) + C) \wedge R(x, y) \leftrightarrow \neg R(x', y) \qquad (*)$$

where $C = |\mathcal{A}|^2 + 1$.

The syntactic congruence of $\mathcal{A}'$ induces an equivalence of finite index on pairs of words $(u, z) \in (\Sigma \cup \{\Box\} \times (\Gamma \cup \{\Box\})^l)^*$. $((u, z) \approx_{\mathcal{A}'} (u', z')$ iff their actions on the states of $\mathcal{A}'$ are identical). Let K be the length of the longest word v such that $(v, \Box^{l|v|})$ is the shortest such representant of its $\approx_{\mathcal{A}'}$-class.

Consider now any x long enough such that $\lceil (|x|+C)\frac{k}{l} \rceil + K < |x|$. During the run of $\mathcal{A}'$ on input (x, y) for any y shorter than $k(|x| + C)$, y will be completely read leaving a suffix v of x, v longer than K, unread. By replacing v with a shorter v' such that $(v, \Box^{l|v|}) \approx_{\mathcal{A}'} (v', \Box^{l|v'|})$ in x we obtain an x' shorter than x, which is by (*) $\sim$-equivalent to x. Thus we have shown that each $\sim$-class has a representant of bounded size, i.e. that there are finitely many such classes as required. ☐

Proposition 8. *For any given $p, q \in \mathbb{N}$ the following problems are undecidable.*

i) Given a rational transduction $R \in \mathsf{Rat}$ is $R \in \mathsf{S}_{(p,q)}\mathsf{Rat}$?
ii) Given a rational transduction $R \in \mathsf{Rat}$ is $R \in \mathsf{SRat}$?

Proof. For i) the proof is essentially the same as for regularity, ii) requires, in addition, a slight adaptation of the technique. Let us therefore give a quick review. Given an instance $I = \{(u_i, v_i) \mid i < n\}$ of PCP consisting of pairs of words over some finite alphabet Γ we define $U = \{(ab^i, u_i) \mid i < n\}$ and $V = \{(ab^i, v_i) \mid i < n\}$. So it is clear that I has a solution iff $W = U^+ \cap V^+ \neq \emptyset$, where U^+ and V^+ are evidently rational. Although the class of rational relations is not closed under complementation, one can show that the complements $\overline{U^+}$ and $\overline{V^+}$ of U^+ and V^+ respectively are in fact rational, hence so is their union $\overline{W} = \overline{U^+} \cup \overline{V^+}$. A number of undecidability results follow from these observations (cf. [2], [10]).

Note that in each pair of U and V the first component ab^i is used only to identify the index of the corresponding second component, their choice is irrelevant as long as they are distinct. Therefore, all of the previous remarks hold, in particular, for $U = U_{\boldsymbol{k}} = \{(ab^{k_i}, u_i) \mid i < n\}$ and $V = V_{\boldsymbol{k}} = \{(ab^{k_i}, v_i) \mid i < n\}$ for any sequence of naturals $\boldsymbol{k} = (k_1, \ldots, k_n)$. In [10] Frougny and Sakarovitch use this fact to show that for an appropriate choice of $\boldsymbol{k}$ $\overline{W}$ is regular iff $W = \emptyset$ iff I has no solution, which is undecidable.

A direct adaptation of their technique proves i). Indeed, for given p, q and instance I of PCP we chose distinct k_i such that $k_i \geq 2\frac{p}{q}\max(|u_i|, |v_i|)$ for all $i < n$. Assume $W = W_{\boldsymbol{k}} \neq \emptyset$. Let $(x, y) \in W$. Then $(x^m, y^m) \in W$ and $|x^m| \geq 2\frac{p}{q}|y^m|$ for any m. It follows from a direct adaptation of Proposition B1 that for any (p, q)-synchronous function f there exists a constant K such that $|q|x| - p|y|| \leq K$ for all $f(x) = y$. Therefore, since W is functional, it is not (p, q)-synchronous, hence, neither is $\overline{W}$. Thus we see that $\overline{W}$ is (p, q)-synchronous iff I has no solution. This concludes the proof of undecidability of i).

To prove undecidability of ii) we give another variant of the previous reduction. Again, let I be a PCP instance over Γ. Let I' be a copy of I over an alphabet Γ' disjoint from Γ. Consider the PCP instance $I \cup I' = \{(u_i, v_i), (u'_i, v'_i) \mid i < n\}$ over $\Gamma \cup \Gamma'$. Let $U = \{(ab^i, u_i) \mid i < n\} \cup \{(a'b'^{2i+1}, u'_i)\}$, $V = \{(ab^i, v_i) \mid i < n\} \cup \{(a'b'^{2i+1}, v'_i)\}$, and $W = U^+ \cap V^+$ as above. If I has no solution then $W = \emptyset$, and if $(i_1, \ldots, i_t)$ is a solution of I with $y = u_{i_1} \cdots u_{i_t} = v_{i_1} \cdots v_{i_t}$ then there are $(x, y) \in W$ and $(x', y') \in W$ such that $|x'| = 2|x|$ and $|y| = |y'|$. Since W is functional, for the same reason as above, it can not be (p, q)-synchronous for any p and q. In other words it is not semi-synchronous, and hence neither is $\overline{W}$. Thus we have shown that the rational $\overline{W}$ is semi-synchronous iff I has no solution, which proves undecidability of ii). □